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Chaos in Quantum Weightless Neuron Node Dynamics

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ABSTRACT

In order to investigate the dynamics of a quantum weightless neuron node we feed its output back as input. Due to the fact that controlled operators used in the neuron circuit usually generate entanglement, we propose a mathematical method to extract the output at time t and build from that output the input at time $t+1$. As a result the time evolution is a real-valued nonlinear map with one real parameter. The dynamics orbits are plotted showing acute sensitivity to initial conditions clearly exhibiting nonlinearity by just looking at amplitude graphs. The fractal geometry and regions of convergence are discussed by their Julia Set images and a new measure for model comparison is put forward.

1. Introduction

We are surrounded by complexity and non-linearity. They emerge from the interactions of systems of either the same or different kinds. Biological systems, weather phenomena, fluid turbulence, radar backscatter from the sea surface, multipath in mobile communication systems and control systems are examples of complex systems. Research in dynamical systems has increased in the past few years in order to understand these systems from initial conditions and their asymptotic behaviour [1,2] with the increase in power of the computer systems.

Poincaré studied the three-body problem when he discovered that small perturbations can significantly affect the solution [3]. Concepts of phase portrait, Poincaré section, periodic orbit, return map, bifurcation and fixed point were first introduced by Poincaré as key descriptive aspects of dynamical systems. The first representation of a chaotic attractor was provided by Edward Norton Lorenz [4] in his attempt to understand weather forecasting through numerical solutions in systems of differential equations.

Since then, important advances in computer graphics, fractals and physics stimulated developments in the field of dynamical systems. Many systems are understood in detail and have been classified into categories according to their number of variables and non-linearity [5].

Chaos in classical neural node [6,7] and networks [8] have been reported in the literature. Evidences for the importance of chaos in natural and artificial brain have been collected in a short survey by Dave Gross in an electronically available article and in the references there [9].

Closed quantum systems are linear (unitary) and the majority of quantum computing literature deals with unitary evolution despite the apparent difficulty of physically isolating quantum systems [10]. In its turn open and measurement based systems can be nonlinear [11–13]. Notwithstanding the traditional unitary approach in quantum computing many studies have been carried out employing nonlinear operators as gates [14]. We should mention that the assumption that a fully quantised system evolution is not sensitive to initial conditions is nevertheless not unanimously accepted and sensitivity to initial conditions of physically realisable fully quantised system has been controversially reported in [15–19]. Measurement of quantum systems affects their dynamics and a non-linear behaviour can emerge from the systems [20,12]. This nonlinear behaviour has serious consequences in the dynamics of the systems bringing chaotic patterns into consideration. Another line of work in this field but not pursued here is the study of quantum systems that are classically chaotic [21].

Some quantum algorithms are naturally iterative but commonly implemented in acyclic circuits subordinated by a classical control. For example the Grover algorithm is the $\Theta(\sqrt{n})$ repetition of the Grover operator G [22]. Grover algorithm can be understood as a set of quantum operators that, through a closed loop, reapplies the output in the input, and the qubits are measured after $\Theta(\sqrt{n})$ times of iteration. Another physical system intrinsically iterative is the control system of a quantum robot [23,24] interacting with the environment for navigation or identification,

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where a quantum computer controls its operations. Despite not being the concern here, when studying cyclic networks of quantum gates is important to comprehend their relation to the halting problem for Turing machines. In acyclic networks of gates it is possible to determine if a algorithm will stop in contrast to cyclic network of arbitrary complexity [25].

Feedback control in quantum computing usually can be performed in the following way. A measurement is performed in some quantum registers and the measurement result is used as feedback. For instance, this strategy is used in [26]. In this paper, we are interested in the alternative method proposed in [27], where it is shown that quantum information in cyclic networks can be beneficial when there is no measurement.

One can understand the dynamics of quantum cyclic networks under the point of view of their operators, its phase analysis, extracting eigenvalues and eigenstates [25]. Studies about weak measurements back into the dynamics of ensemble of quantum systems were presented by Lloyd and Slotine [28]. Conditional dynamics of qubits iterated by a unitary operator coupled with a measurement-induced nonlinearity is investigated by Kiss et al. [20] and shown to be exponentially sensitive to initial condition with positive Lyapunov demonstrating chaotic behaviour. The nonlinear operator employed arises in quantum state purification protocols where the nonlinear effects can guarantee the unconditional security of quantum cryptographic key distribution protocols. Quantum systems that interact with an environment through measurement can be chaotic and nonlinear [29–33].

In this work we show a set of experiments and analysis of the dynamics of the qRAM and $|\psi\rangle$-RAM quantum weightless neuron nodes which demonstrate high degree of sensitivity to the initial conditions and chaotic behaviour. For that, we have used the quantum operator of the respective node and a measurement induced nonlinear step. After a short review of the basic notions of Quantum Computing (Section 2), Dynamical Systems (Section 3), Classical Weightless Neural Networks (Section 4) and Quantum Weightless Neural Networks (Section 5) the proposed Models of Dynamics (Section 6) is presented where is given a proof that the target qubits after a generic controlled unitary operator cannot always be decomposable as a product of two isolated quantum states, *i.e.* they are entangled, Theorem 6.1. In Section 6.2.1 the method for mathematically extract the amplitudes of a (possibly) entangled states is presented while in Section 6.2.2 the experiments are explained and analysed. A mathematical procedure to recover the amplitudes of the output qubit is discussed. It is observed high sensitivity to initial conditions and chaotic behaviour. After that, the results are analysed under the perspective of Amplitude Graph and a quantitative study is introduced by a measure of variation.

2. Quantum computing

One *quantum bit* (qubit) is a two-dimensional vector in the complex vector space $\mathbb{C}^2$. Any qubit $|\psi\rangle$ can be written as a linear combination of vectors (or states) of $\mathbb{C}^2$ canonical (or computational) basis $|0\rangle = [1,0]^T$ and $|1\rangle = [0,1]^T$ as viewed in the following equation:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{1}$$

where α and β are complex numbers and $|\alpha|^2 + |\beta|^2 = 1$.

Tensor product $\otimes$ is used to represent quantum systems with two or more qubits $|ij\rangle = |i\rangle \otimes |j\rangle$. Let A and B be two vector spaces the tensor product of A and B, denoted by $A \otimes B$, is the vector space generated by the tensor product of all vectors $|a\rangle \otimes |b\rangle$, with $|a\rangle \in A$ and $|b\rangle \in B$. Some states $|\psi\rangle \in A \otimes B$ cannot be written as a product of states of its component systems A and B. States with this property are called *entangled* states, for instance two entangled qubits are the Bell states described in the following equation:

$$\begin{aligned} |\Phi^+\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\ |\Phi^-\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\ |\Psi^+\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\ |\Psi^-\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}} \end{aligned} \tag{2}$$

Quantum operator $\mathbf{U}$ over n qubits is a unitary complex matrix of order $2^n \times 2^n$. For example, some operators over 1 qubit are Identity $\mathbf{I}$, NOT $\mathbf{X}$ and Hadamard $\mathbf{H}$, described below in Eqs. (3) and (4) in matrix form and operator form. The combination of these unitary operators forms a quantum circuit.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} \mathbf{I}|0\rangle = |0\rangle \\ \mathbf{I}|1\rangle = |1\rangle \end{matrix}, \quad \mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{matrix} \mathbf{X}|0\rangle = |1\rangle \\ \mathbf{X}|1\rangle = |0\rangle \end{matrix} \tag{3}$$

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{matrix} \mathbf{H}|0\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle) \\ \mathbf{H}|1\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle) \end{matrix} \tag{4}$$

The Identity operator $\mathbf{I}$ generates the output exactly as the input; $\mathbf{X}$ operator works as the classic NOT in the computational basis; Hadamard $\mathbf{H}$ generates a superposition of states when applied in a computational basis. The CNOT operator has 2 inputs and 2 outputs and flips the second one if the first is 1, as shown in Fig. 1.

The operators of quantum computation can be seen as special kinds of linear transformations, as matrices that operates in a vector basis. These special matrices are unitary and invertible [34].

3. Dynamical systems

Systems that have variation in time can be usually dealt with mathematical structures having time as parameter. This time iterative process is the subject of the field Dynamical Systems where there are many tools and concepts that help designers and engineers to investigate the temporal behaviour of systems. Some of these concepts are presented in this section to help understanding and evaluating the models investigated in this work.

3.1. Orbits

There are many problems in Science in general and in Mathematics in particular that involve iteration [5]. Iteration means to repeat a process many times. In dynamics the process that is repeated is the application of a function. The result of the application of a function in previous time is used as input in the same function in the current time.

Given $x_0 \in \mathbf{R}$, we define the orbit of x_0 under F to be the sequence of points $x_0, x_1 = F(x_0), x_2 = F(x_1), \cdots x_n = F(x_{n-1}) \cdots$. The point x_0 is called "seed" of the orbit.

Sometimes it is useful to deal with a family of functions parametrised by a constant and so it is normal to represent it as $F_c(z)$ where c is a constant. As example, we have $F_c(z) = z^2 + c$, and $F_2(z) = z^2 + 2$, where $c=2$. This representation helps us to categorise these families of functions.

3.2. Julia Set

Julia Set is the place where every chaotic behaviour of a complex function occurs [35]. For example, the squaring map $Q_0(z) = z^2$ is chaotic in the unit circle, because if $|z| < 1$ then $|Q_0^n(z)| \to 0$,