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Tensor completion via multi-shared-modes canonical correlation analysis ☆

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ABSTRACT

Low-rank tensor completion (LRTC) has been applied in many real-world problems. But most of the existing LRTC methods recover a tensor on a single dataset with the low-rank assumption, suffering from a low accuracy due to the complicated structures of higher-order data. To address this issue, we propose a novel tensor completion method for two correlated tensor datasets obtained from different sources. We first introduce the correlated tensors with multiple shared modes via tensor canonical correlation analysis (TCCA), and reveal the relationship between the transformation matrices of TCCA and the Tucker decomposition. Then we develop a Tucker-*n* decomposition method with *n* invariant modes to capture the latent structures of incomplete tensors, in which sufficient discriminative information for TCCA can be flexibly maintained by varying the number of invariant modes. Finally, we combine the Tucker-*n* decomposition method for LRTC with the correlation of TCCA as a regularizer to improve the completion performance, and derive relative error bounds for our LRTC approach to guarantee the recovery accuracy. Experimental results on synthetic and real data demonstrate the accuracy and efficiency of the proposed approach, as well as the benefit of multiple shared modes.

1. Introduction

Tensors, as generalizations of vectors and matrices, provide a natural and efficient representation of multi-dimensional data including visual, spatiotemporal and neuroimaging data [1–4]. Meanwhile, tensor data with missing entries arise in many practical problems due to loss of information, errors in the data collection process, and costly experiments [5]. The missing entries recovery depends on the relationship between the observed entries and the unknown ones. Since tensor decompositions give a concise representation of the underlying structures and correlations of the tensor data, they have been used as common tools for recovering incomplete tensor [4,6–8].

Recently, low-rank matrix completion has been actively studied [9,10]. The nuclear norm is usually used to approximate the rank function of matrices and the resulting optimization problems are convex. Although there are obstacles to generalize the methods for matrix completion to tensor recovery since most tensor problems are NP-hard [11], several tensor completion problems have been formulated using extensions of nuclear norm [8]. In [4], tensor completion problem based on a novel definition of the nuclear norm was presented and three convex optimization algorithms, SiLRTC, FaLRTC, and HaLRTC, were proposed to tackle the problem. Besides, many algorithms have been designed to alleviate the estimation bias of the tensor nuclear norm based formulation for low-rank tensor completion (LRTC) [12,13]. Several LRTC methods with low-rank assumption have also been successfully applied to electroencephalo-graph (EEG) data analysis and natural images with less computational cost [14,15]. The CP decomposition was used to formulate a weighted least squares problem for LRTC, in which the tensor rank was supposed known [6]. Another well-known tensor decomposition, Tucker decomposition, was also introduced to formulate the Tucker weighted optimization method for LRTC, in which the rank of input tensor can be overestimated or underestimated [7].

However, only one tensor dataset was considered in the most existing LRTC methods, while different correlated datasets are usually obtained simultaneous from multiple sources of the same object. In the case of tensor datasets from multiple sources, tensor decompositions have been introduced to solve multi-relational learning and data fusion [16,17]. Two regularization approaches for tensor completion were proposed using graph Laplacians induced from the relationships among datasets [18]. Recently, some multi-tensor completion methods were proposed for estimating missing values in video data [19,20].

For multiple datasets, canonical correlation analysis (CCA) is a powerful unsupervised tool for discovering relationships between

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two sources of information [21]. Under the multilinear subspace learning (MSL) framework, the CCA methods were extended to the multilinear case via tensor-to-vector projection [22,23]. Another tensor CCA (TCCA) was proposed for third-order tensors with shared modes [1]. Recently, TCCA has been widely used in the context of hand gesture recognition, action categorization and functional magnetic resonance imaging (fMRI) [24,25], in which the correlations of high-order data from different sources were measured by TCCA. Thus, in addition to the low-rank assumption, we assume that the tensors from two different sources have the correlation, and introduce more prior information for tensor completion by a TCCA regularization term.

In this paper, we will present a novel method to complete tensor data using TCCA with multiple shared modes, in which incomplete tensor and its correlated tensor are considered as inputs. In other words, we find two sets of orthogonal canonical transformations with n invariant modes, by which two tensor datasets are projected into low-dimensional spaces to maximize their correlation. Then, the optimal orthogonal canonical transformations can be used to recover the incomplete tensor data.

The rest of the paper is organized as follows. In Section 2, we provide the notations and basic facts about tensor algebra, as well as a concise introduction to TCCA and LRTC. A method to recover incomplete tensors via TCCA is proposed in Section 3. Section 4 presents the performance of our algorithm on synthetic and real tensor data. We end in Section 5 with conclusion.

2. Notations and background

This section introduces some notations and definitions used throughout the paper and reviews the related works in TCCA and LRTC.

2.1. Tensor algebra

For convenience, we denote vectors as boldface lowercase letters, i.e., $\boldsymbol{x}$, and matrices as boldface capital letters, i.e., $\boldsymbol{X}$. N-th order tensors, which we denote by calligraphic letters, e.g., $\mathcal{X}$, are higher-order generalizations of vectors (first-order tensors) and matrices (second-order tensors). More generally, the order N of a tensor is the number of modes. Entries of a tensor are denoted by lowercase letters with subscripts, i.e., the $(i_1, i_2, \dots, i_N)$ entry of an N-th order tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is denoted by $x_{i_1, i_2, \dots, i_N}$.

Definition 2.1 (*n-Mode unfolding and n-rank*). An n-mode vector of $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is an element of $\mathbb{R}^{I_n}$ obtained from $\mathcal{X}$ by varying the index i_n and keeping the other indices fixed. The n-mode unfolding or matricization of a tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is the matrix $\mathcal{X}_{(n)} \in \mathbb{R}^{I_n \times L_n}$ whose columns are the n-mode vectors, where

$$L_n = \prod_{k \in \{1,2,\dots,n-1,n+1,\dots,N\}} I_k.$$

Then, the n-rank of $\mathcal{X}$, denoted $\mathrm{rank}_n(\mathcal{X})$, is the rank of $\mathcal{X}_{(n)}$. If let $r_n = \mathrm{rank}_n(\mathcal{X})$ for $n = 1, 2, \dots, N$, we can say that $\mathcal{X}$ is a rank-$(r_1, r_2, \dots, r_n)$ tensor.

Definition 2.2 (*n-Mode product*). The n-mode product of a tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ with a matrix $\boldsymbol{U} \in \mathbb{R}^{J \times I_n}$ is denoted by $\mathcal{X} \times_n \boldsymbol{U}$ with size $I_1 \times I_2 \times \cdots \times I_{n-1} \times J \times I_{n+1} \times \cdots \times I_N$, defined by $(\mathcal{X} \times_n \boldsymbol{U})_{(n)} = \boldsymbol{U}\mathcal{X}_{(n)}$.

Definition 2.3 (*Tucker decomposition*). The Tucker decomposition decomposes a tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ into a core tensor $\mathcal{G} \in \mathbb{R}^{J_1 \times J_2 \times \cdots \times J_N}$ multiplied by a matrix along each mode:

$$\mathcal{X} = \mathcal{G} \times_1 \boldsymbol{U}_1 \times_2 \boldsymbol{U}_2 \times_3 \cdots \times_N \boldsymbol{U}_N,$$

where $\{\boldsymbol{U}_n\}$ are factor matrices (usually orthogonal). Additionally, $J_k = r_k$ for $k = 1, 2, \dots, N$ if $\mathcal{X}$ is rank-$(r_1, r_2, \dots, r_n)$.

Definition 2.4 (*Frobenius norm*). The Frobenius norm of tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is denoted by $\|\mathcal{X}\|_F$ and defined as

$$\|\mathcal{X}\|_F^2 = \langle \mathcal{X}, \mathcal{X} \rangle = \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} x_{i_1, i_2, \dots, i_N}^2.$$

2.2. Tensor canonical correlation analysis (TCCA)

In [1,25], for two third-order tensors data $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{R}^{I \times J \times K}$, I single-shared-mode TCCA for third-order tensor is described as follows with respect to the canonical correlation ρ:

$$\rho = \max_{\boldsymbol{U}_j^{(1)}, \boldsymbol{U}_k^{(1)}, \boldsymbol{U}_j^{(2)}, \boldsymbol{U}_k^{(2)}} \langle \mathcal{X}_1 \times_j \boldsymbol{U}_j^{(1)\top} \times_k \boldsymbol{U}_k^{(1)\top}, \mathcal{X}_2 \times_j \boldsymbol{U}_j^{(2)\top} \times_k \boldsymbol{U}_k^{(2)\top} \rangle,$$

where $\boldsymbol{U}_j^{(1)}, \boldsymbol{U}_k^{(1)}, \boldsymbol{U}_j^{(2)}, \boldsymbol{U}_k^{(2)}$ are orthogonal sets of canonical transformations.

Furthermore, IJ joint-shared-mode TCCA [1,25] for third-order tensor $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{R}^{I \times J \times K}$ is described as follows:

$$\rho = \max_{\boldsymbol{U}_k^{(1)}, \boldsymbol{U}_k^{(2)}} \langle \mathcal{X}_1 \times_k \boldsymbol{U}_k^{(1)\top}, \mathcal{X}_2 \times_k \boldsymbol{U}_k^{(2)\top} \rangle,$$

where orthogonal canonical transformations $\boldsymbol{U}_k^{(1)}, \boldsymbol{U}_k^{(2)}$ are found to maximize the inner product of the canonical objects.

2.3. Low-rank tensor completion (LRTC)

In Filipović et al.'s work, the Tucker weighted optimization (Tucker–Wopt) for low-rank tensor completion is given as follows which can be solved by nonlinear conjugate gradient method [7]:

$$\min_{\mathcal{G}, \{\boldsymbol{U}_n\}} \|\mathcal{W} \odot (\mathcal{X} - \mathcal{G} \times_1 \boldsymbol{U}_1 \times_2 \cdots \times_N \boldsymbol{U}_N)\|_F^2,$$

where the symbol $\odot$ denotes the Hadamard (elementwise) product and $\mathcal{W}$ is a nonnegative weight tensor with the same size as $\mathcal{X} \in R^{I_1 \times I_2 \times \cdots \times I_N}$:

$$w_{i_1, i_2, \dots, i_N} = \begin{cases} 1 & \text{if } x_{i_1, i_2, \dots, i_N} \in \Omega, \\ 0 & \text{if } x_{i_1, i_2, \dots, i_N} \in \Omega^C, \end{cases}$$

where Ω denotes the set of indexes of known elements and Ω_C denotes the complement of Ω.

3. Low-rank tensor completion on correlated data

In this section, we will describe our low-rank tensor completion method with high-order canonical correlation analysis in detail, in which two tensor datasets are considered as inputs.

3.1. Multi-shared-modes TCCA (MCCA)

In order to make use of the correlation of two tensor datasets, we firstly generalize the TCCA of two third-order tensors into that of two high-order tensors with multiple shared modes. Then, MCCA is given as follows in which input $\mathcal{X}_1, \mathcal{X}_2$ are described as high-order tensors with M shared modes:

$$\rho = \max_{\{\boldsymbol{U}_n^{(k)}\}} \langle \mathcal{X}_1 \times_{M+1} \boldsymbol{U}_{M+1}^{(1)\top} \cdots \times_N \boldsymbol{U}_N^{(1)\top}, \ \mathcal{X}_2 \times_{M+1} \boldsymbol{U}_{M+1}^{(2)\top} \cdots \times_N \boldsymbol{U}_N^{(2)\top} \rangle, \quad (1)$$

where

$$\mathcal{X}_1 \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_M \times I_{M+1} \times \cdots \times I_N}, \ \mathcal{X}_2 \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_M \times J_{M+1} \times \cdots \times J_N},$$

and orthogonal sets of canonical transformations are denoted as

$$\boldsymbol{U}_n^{(1)} = (\boldsymbol{u}_{1n}^1, \boldsymbol{u}_{1n}^2, \dots, \boldsymbol{u}_{1n}^{r_n}) \in \mathbb{R}^{I_n \times r_n},$$