Neural Networks 79 (2016) 117–127

Contents lists available at ScienceDirect

Neural Networks

journal homepage: www.elsevier.com/locate/neunet

Multistability analysis of a general class of recurrent neural networks with non-monotonic activation functions and time-varying delays

Peng Liu [a,b], Zhigang Zeng [a,b], Jun Wang [c,*]

[a] School of Automation, Huazhong University of Science and Technology, Wuhan 430074, China

[b] Key Laboratory of Image Processing and Intelligent Control of Education Ministry of China, Wuhan 430074, China

[c] Department of Computer Science, College of Science and Engineering, City University of Hong Kong, Kowloon Tong, Hong Kong

ARTICLE INFO

Article history:
Received 1 September 2015
Received in revised form 21 March 2016
Accepted 29 March 2016
Available online 12 April 2016

Keywords:
Cohen–Grossberg neural networks
Multistability
Non-monotonic activation functions
Time-varying delays

ABSTRACT

This paper addresses the multistability for a general class of recurrent neural networks with time-varying delays. Without assuming the linearity or monotonicity of the activation functions, several new sufficient conditions are obtained to ensure the existence of $(2K + 1)^n$ equilibrium points and the exponential stability of $(K + 1)^n$ equilibrium points among them for n-neuron neural networks, where K is a positive integer and determined by the type of activation functions and the parameters of neural network jointly. The obtained results generalize and improve the earlier publications. Furthermore, the attraction basins of these exponentially stable equilibrium points are estimated. It is revealed that the attraction basins of these exponentially stable equilibrium points can be larger than their originally partitioned subsets. Finally, three illustrative numerical examples show the effectiveness of theoretical results.

1. Introduction

A general class of recurrent neural networks, proposed by Cohen and Grossberg (1983), have received extensive attention due to their widespread applications in various areas. Different applications may rely on different underlying dynamic behaviors of neural networks. For many applications, especially in classification, associative memory, pattern recognition, it is essential to determine the existence of multiple equilibrium points and their locally stable properties. Thus the multistability analysis of neural networks has been an active area of research in recent years.

There are many remarkable results that have been presented for multistability of neural networks, including Cohen–Grossberg ones. In Zeng, Wang, and Liao (2004), the authors studied on the multistability of recurrent neural networks through linearly partitioning the state space, which showed that the $N \times M$-neuron recurrent neural networks with one step piecewise linear activation function can have $3^{N\times M}$ equilibrium points and $2^{N\times M}$ of them located in saturation regions were locally exponentially stable. In Cheng, Lin, and Shih (2006), the authors investigated Hopfield-type neural networks with sigmoidal activation functions and presented that the n-neuron Hopfield-type neural networks can have 3^n equilibrium points and 2^n of them were asymptotically stable. Cao, Feng, and Wang (2008) investigated multistability and multiperiodicity of delayed Cohen–Grossberg neural networks with nondecreasing piecewise linear activation functions based on the Cauchy convergence principle. Huang and Cao (2010) presented a delay-dependent multistability criterion for recurrent neural networks with sigmoidal activation functions by constructing Lyapunov functional and using matrix inequality techniques, in which the obtained results are flexible and conservative. In Wang and Chen (2012), the authors focused on the multistability of a class of neural networks with Mexican-hat-type activation functions, which were piecewise linear functions actually.

It should be noted that more locally stable equilibrium points imply more storage capacity of neural networks. Moreover, the number of locally stable equilibrium points is closely related to geometrical configuration of activation functions. Hence it is reasonable to construct some suitable activation functions to increase the number of locally stable equilibrium points. In Wang, Lu, and Chen (2010), Zeng and Zheng (2013), the neural networks with a class of nondecreasing piecewise linear activation functions with $2r$ corner points were investigated. It was revealed that the n-neuron dynamical systems can have $(2r + 1)^n$ equilibria under some conditions, with $(r + 1)^n$ of them being locally exponentially stable. In Wang and Chen (2014), the paper is concerned with the problem of μ-stability of recurrent neural networks with K-level

* Corresponding author.
E-mail addresses: liupengjz@163.com (P. Liu), hustzgzeng@gmail.com (Z. Zeng), jwang.cs@cityu.edu.hk (J. Wang).

http://dx.doi.org/10.1016/j.neunet.2016.03.010
0893-6080/

nonlinear monotone activation functions and unbounded time varying delays, in which they could store $(2K+1)^n$ equilibrium points, $(K+1)^n$ of them are locally μ-stable. Similar results were presented in Rakkiyappan, Velmurugan, and Cao (2015). There are many other valuable works on multistability of neural networks, please refer to Cheng, Lin, Shih, and Tseng (2015), Cheng and Shih (2009), Di Marco, Forti, Grazzini, and Pancioni (2012, 2014), Forti (2002), Forti and Tesi (2001), Kaslik and Sivasundaram (2011), Nie and Cao (2011), Song and Cao (2007), Wen, Huang, Zeng, Chen, and Peng (2015), Zeng, Huang, and Zheng (2010), Zeng and Wang (2006), Zeng and Zheng (2012) and the references cited therein.

To the best of our knowledge, in the existing literature (including the works mentioned above) on multistability analysis, the activation functions are usually chosen to be either nondecreasing or piecewise linear. But there are also many other notable types of neural networks with both nonlinear and non-monotonic activation functions requiring that neural networks have multiple equilibrium points and their corresponding locally multistability. For example, the Self-Organizing Map (SOM, commonly also known as Kohonen network), widely used in various fields especially in associative memory (Kohonen, 1989, 2001; Kosko, 1988), chooses nonlinear Mexican hat functions as its activation functions. Radical basis function (RBF) networks, which can be applied in classification, generally use Gaussian activation functions (Orr, 1996). The trigonometric function (McCaughan, 1997; Nakagawa, 1998), the Morita function (Morita, 1993), the Crespi function (Crespi, 1999) and so on as activation functions used in neural networks also share the common feature that they are all nonlinear and non-monotonic. Furthermore, as pointed in Crespi (1999), Morita (1993), Obayashi, Omiya, Kuremoto, and Kobayashi (2008), Yoshizawa, Morita, and Amari (1993), the associative ability of neural networks can be improved significantly by using non-monotonic, rather than monotonic, functions as activation functions. Hence, it is necessary for us to study the multistability of neural networks with non-monotonic activation functions.

With the motivations illustrated as above, our main objective in this paper is to investigate the multistability of Cohen–Grossberg neural networks with non-monotonic activation functions and time-varying delays. We derive some sufficient conditions under which the n-neuron Cohen–Grossberg neural networks with non-monotonic activation functions and time-varying delays can have $(2K+1)^n$ equilibrium points, among which $(K+1)^n$ equilibrium points are locally exponentially stable, and also estimated the attraction basins of these exponentially stable equilibrium points.

The remaining part of this paper is organized as follows. In Section 2, backgrounds about the neural network model and state-space partition are given, then preliminary results in form of lemmas for ascertaining the existence of multiple equilibrium points of neural networks and positive invariance of partitioned subsets are derived. The main results on the exponential stability of multiple equilibrium points of Cohen–Grossberg neural networks with non-monotonic activation functions and time-varying delays are presented in Section 3. In Section 4, the attraction basins of multiple equilibrium points are estimated. In Section 5, three numerical examples are given. Finally, some concluding remarks are made in Section 6.

2. Model descriptions and preliminaries

Consider a class of Cohen–Grossberg neural network with time-varying delays described as follows,

$$\frac{dx_i(t)}{dt} = -a_i(x_i(t))\Big[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}f_j(x_j(t)) - \sum_{j=1}^{n} d_{ij}f_j(x_j(t-\tau_j(t))) - I_i\Big], \tag{1}$$

where $i = 1, \ldots, n$, $x = (x_1, \ldots, x_n)^T \in \Re^n$ is the state vector. $a_i(x_i(t))$ and $b_i(x_i(t))$ represent the amplification and the self-signal functions, respectively, $C = (c_{ij})_{n\times n}$ and $D = (d_{ij})_{n\times n}$ denote the normal and the delayed connection weight matrices, respectively, $f_i(\cdot)$ is a neuron activation function, I_i is a constant external input. $\tau_i(t)$ corresponds to the transmission delay and satisfies $0 \le \tau_i(t) \le \tau$ (τ is a constant).

Let $C([-\tau, 0], \mathcal{D})$ be the Banach space of functions mapping $[-\tau, 0]$ into $\mathcal{D} \subset \Re^n$ with norm defined by $\|\phi\|_\tau = \max_{1\le i\le n}\{\sup_{s\in[-\tau,0]} |\phi_i(s)|\}$, where $\phi(s) = (\phi_1(s), \ldots, \phi_n(s))^T \in \Re^n$. The initial condition of (1) can be represented as $\phi(s) \in C([-\tau, 0], \Re^n)$, that is

$$x_i(s) = \phi_i(s), \quad s \in [-\tau, 0],\ i = 1, 2, \ldots, n. \tag{2}$$

We suppose that the parameters of neural network (1) and activation functions in this paper satisfy the following assumptions, respectively.

(A1) Each $a_i(u)$ is continuous and there exist positive constants $\underline{a}_i$ and $\bar{a}_i$ such that $\underline{a}_i \le a_i(u) \le \bar{a}_i$, for $u \in \Re$, $i = 1, \ldots, n$.

(A2) $b_i(\cdot)$ and its inverse function $b_i^{-1}(\cdot)$ are locally Lipschitz continuous and there exists $\gamma_i > 0$ such that $u[b_i(u+r) - b_i(r)] \ge \gamma_i u^2$, for all $u, r \in \Re$, $i = 1, \ldots, n$.

(A3) The activation functions $f_i(u)$ $(i = 1, \ldots, n)$ are continuous and there exist constants $m_i < M_i$ such that for $i = 1, \ldots, n$,

$$m_i \le f_i(u) \le M_i, \quad \text{for any } u \in \Re.$$

(A4) There exist constants

$$-\infty \le q_i^{(0)} < p_i^{(0)} < q_i^{(1)} < p_i^{(1)} < \cdots < q_i^{(K-1)} < p_i^{(K-1)} < q_i^{(K)} < p_i^{(K)} \le +\infty,$$

$\lambda_i^{(k)}$, $\mu_i^{(l)}$ and nonnegative constants $\eta_i^{(k)}$, $\nu_i^{(l)}$ $(k = 0, 1, \ldots, K;\ l = 1, \ldots, K;\ K$ is a positive integer) such that for $i = 1, \ldots, n$

$$\lambda_i^{(k)} \le \frac{f_i(u) - f_i(v)}{u - v} \le \eta_i^{(k)}, \quad \text{for any } u, v \in (q_i^{(k)}, p_i^{(k)}),\ k = 0, 1, \ldots, K,$$

$$\mu_i^{(l)} \le \frac{f_i(u) - f_i(v)}{u - v} \le \nu_i^{(l)}, \quad \text{for any } u, v \in [p_i^{(l-1)}, q_i^{(l)}],\ l = 1, \ldots, K.$$

Remark 1. Assumptions (A1)–(A3) ensure the boundedness of the solutions of the neural network (1), which are hypothesized in many references dealing with the stability of neural networks (Chen & Rong, 2004; Liao, Li, & Wong, 2004; Ye, Michel, & Wang, 1995). Moreover, Assumption (A1) also guarantees the invariance of the inequality direction when investigating the stability of the equilibrium points.

Remark 2. It should be noted that Assumption (A4) implies that each activation function is Lipschitz continuous with different Lipschitz constants in different intervals. This is necessary owing to the fact that the multistability analysis of neural networks is quite different from mono-stability analysis. In mono-stability analysis, the equilibrium point is unique and its global stability is considered, and usually we just assume that each activation function is Lipschitz with one Lipschitz constant. In contrast in multistability analysis, we should determine the number of the equilibrium points first, which are closely related to the type of activation functions, then we analyze the multistability and attractive domains of some of these points. Hence, the state space needs to be partitioned into several subsets, and the properties of activation functions are generally assumed to vary with different partitioned intervals in