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Stability and bifurcation analysis of two-neuron network with discrete and distributed delays

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ABSTRACT

In this paper, we give a detailed Hopf bifurcation analysis of a recurrent neural network system involving both discrete and distributed delays. Choosing the sum of the discrete delay terms as a bifurcation parameter the existence of Hopf bifurcation is demonstrated. In particular, the formulae determining the direction of the bifurcations and the stability of the bifurcating periodic solutions are studied by using the normal form theory and center manifold theorem. Finally, numerical simulations supporting our theoretical results are also included.

1. Introduction

Recently, research of recurrent neural networks (RNNs) (especially Hopfield neural networks) has attracted the attention of great number of investigators [8,10,13,20,22,31,32,33]. Major contributions to the theory and design of neural networks dates back to the 1980s [13]. Hopfield [14] has been a pioneer study in recurrent networks with symmetric synaptic connections using energy function. Based on the Hopfield network model, Babcock and Westervelt [2] have studied the dynamics of simple electronic neural networks and analyzed two neurons with discrete time delays. Marcus and Westervelt [18] have considered the stability of analog neural networks with delayed response and showed that continuous time analog networks can exhibit sustained oscillation when time delay is included. Afterwards, the dynamic properties of neural networks have been studied very much in the literature [3,7,13,21,23,29].

RNNs which generate stable oscillations have been used to model certain biological phenomena (see [28] and the references therein). Physiological experiments suggest that brain has chaotic structure. If this chaotic behaviour changes, as in Alzheimer disease, the brain could be slower on rapid state transitions essential for information processing [9]. For epilepsy disease, it has been assumed that the stabilization of unstable patterns in the healthy chaotic brain dynamics is the cause of the increased rhythmicity observed in EEG recordings at the onset of epileptic seizures. In this sense, neural networks are important for controlling chaotic dynamical systems. If RNNs have periodic orbits, then such periodic orbits are meant to capture the idea that certain activities or motions are learned by repetition [28].

It is known that time delay may affect dynamical behaviors of neural networks [1,27]. The delayed axonal signal transmissions in neural networks make the dynamic behaviors more complicated and may destabilize stable equilibria, and give rise to periodic oscillation, bifurcation and chaos [10]. Therefore, stability analysis of neural networks with constant or time-varying delays has been an attractive subject of research. Although delay differential equations present richer dynamics than simple low dimensional differential equations, qualitative analysis of delay differential equations is much more complex [4,5,15].

Bifurcation is one of the most important dynamical phenomena for the nonlinear neural systems [26]. Changing control parameter may cause instability, losing equilibrium points or occurring periodic solutions. It is determined locally whether a system has periodic solutions via Hopf bifurcation theory. In bifurcation

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theory literature, choosing the bifurcation parameter as delay parameter is common to see the effect of delay term.

Neural networks with delays have very rich and complex dynamics. There are some other methods to investigate limit cycles or periodic orbits in RNNs [28]. Using Hopf bifurcation theorem is one of the widespread techniques. Since determination of the roots of the characteristic polynomial corresponding to linearized system and then applying normal form theory are rather arduous, researchers have paid attention to study the dynamics of small-scale neurons rather than large-scale networks to understand these complicated structures.

As pointed out by Ruan and Filfil [21], neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Hence, there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays. In these circumstances the signal propagation is not instantaneous and cannot be modeled with discrete delays. Therefore, a more appropriate way is to incorporate continuously distributed delays in finite or/and infinite time. Also, in some cases, the entire history affects the current state, so considering infinite time delay is more general [16,24,30]. Moreover, a distributed delay becomes a discrete delay when the delay kernel is a Dirac delta function at a certain time. We refer to [6,10,17,21,34] and the references therein for related work on networks with distributed delays.

Olien and Bélair [19] investigated the stability of a two-neuron system with discrete time delays, that is,

$$u_1'(t) = -u_1(t) + a_{11} f(u_1(t-\tau_1)) + a_{12} f(u_2(t-\tau_2)),$$
$$u_2'(t) = -u_2(t) + a_{21} f(u_1(t-\tau_1)) + a_{22} f(u_2(t-\tau_2)). \quad (1)$$

They studied several cases, including $\tau_1 = \tau_2$, $a_{11} = a_{22}$. They found that system (1) may undergo Hopf bifurcations at certain values of delay. Following this work, properties of two-neuron networks with delays have been studied intensively [6,11,21,23,29,32,34]. In the case of multiple delays, the dynamics of systems could be more complex and interesting since the characteristic equation is transcendental. Recently, Li and Hu [17] studied the following differential equations with multiple delays:

$$x_1'(t) = -x_1(t) + a_{11} f\left(\int_{-\infty}^{t} F(t-s) x_1(s)\, ds\right) + a_{12} f(x_2(t-\tau)),$$
$$x_2'(t) = -x_2(t) + a_{21} f(x_1(t-\tau)) + a_{22} f\left(\int_{-\infty}^{t} F(t-s) x_2(s)\, ds\right). \quad (2)$$

First, they investigated the stability of the zero equilibrium using Routh–Hurwitz criterion when delay term $\tau = 0$. And then taking discrete delay τ as bifurcation parameter, they showed the existence of local Hopf bifurcation using Hopf bifurcation theorem conditions.

In this paper, extending the idea above, we take discrete delays between neurons that appears in Eq. (2) differently and consider the following general two-neuron network with discrete and distributed delays, that is,

$$x_1'(t) = -x_1(t) + a_{11} f_{11}\left(\int_{-\infty}^{t} F(t-s) x_1(s)\, ds\right) + a_{12} f_{12}(x_2(t-\tau_2)),$$
$$x_2'(t) = -x_2(t) + a_{21} f_{21}(x_1(t-\tau_1)) + a_{22} f_{22}\left(\int_{-\infty}^{t} F(t-s) x_2(s)\, ds\right), \quad (3)$$

where $x_i'(t) = dx_i/dt$, $x_i(t)$ represents the state of the ith neuron at time t and a_{ij} ($i=1,2$ and $j=1,2$) are real constants. Here, $F(.)$ is nonnegative bounded delay kernel defined on $[0, \infty)$ which reflects the influence of the past states on the current dynamics. System (3) is reduced to system (2) if $f_{ij} = f$, $i=1,2$ and $j=1,2$ and $\tau_1 = \tau_2$. Similarly, it is reduced to system (1) when the delay kernel is taken as Dirac delta function and $f_{ij} = \tanh$ (see [17]). The architecture of system (3) is illustrated in Fig. 1. One should underline that system (3) can maintain a periodic orbit such that these periodic orbits present periodic pattern and have been used in learning theory, which are meant to capture the idea that certain activities or motions are learned by repetition [28].

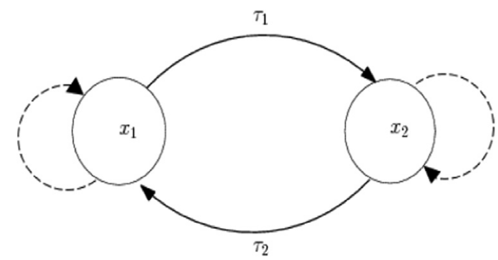

Fig. 1. Architecture of the model (3). Two neurons send signals to each other with a discrete delay, τ_j, $j=1,2$. One element receives one delayed input from itself with distributed delay, that has been shown with dashed line.

Our aim in this paper is to give a detailed Hopf bifurcation analysis of Eq. (3) with second distributed delay is omitted as the first case. A complete Hopf bifurcation analysis of Eq. (3) will be studied in a separate paper later since the dimension of the system is getting higher and characteristic equation is more complex. Choosing $\tau = \tau_1 + \tau_2$ as a bifurcation parameter we study the stability of the zero solution and investigate the local Hopf bifurcation properties.

This paper is organized as follows. In Section 2, stability of the equilibrium and the existence of Hopf bifurcation are investigated. In Section 3, the direction of Hopf bifurcation and the stability and period of bifurcating periodic solutions on the center manifold are determined. Finally, in Section 4, we consider an example and simulate it using MATLAB to support our theoretical results.

2. Stability analysis and Hopf bifurcation

As we mentioned it in Introduction, we first study a simplified version of Eq. (3) in this paper. This model that we consider has the following form:

$$x_1'(t) = -x_1(t) + a_{11} f_{11}\left(\int_{-\infty}^{t} F(t-s) x_1(s)\, ds\right) + a_{12} f_{12}(x_2(t-\tau_2))$$
$$x_2'(t) = -x_2(t) + a_{21} f_{21}(x_1(t-\tau_1)) + a_{22} f_{22}(x_2(t)). \quad (4)$$

In order not to affect the equilibrium values, we normalize the kernel such that $\int_0^\infty F(s)\, ds = 1$. In this paper, we consider only the weak kernel, that is,

$$F(s) = \alpha e^{-\alpha s}, \quad \alpha > 0,$$

where α reflects the mean delay of the weak kernel. Now, it is necessary to make the following assumptions:

(H1) $f_{ij} \in C^3$, $f_{ij}(0) = 0$, $(i = 1,2 \text{ and } j = 1,2)$,
(H2) $\tau = \tau_1 + \tau_2$.

For convenience, we define a new variable as follows:

$$x_3(t) = \int_{-\infty}^{t} F(t-s) x_1(s)\, ds.$$

Then by the linear chain trick technique, system (4) can be transformed into the following system:

$$x_1'(t) = -x_1(t) + a_{11} f_{11}(x_3(t)) + a_{12} f_{12}(x_2(t-\tau_2)),$$
$$x_2'(t) = -x_2(t) + a_{21} f_{21}(x_1(t-\tau_1)) + a_{22} f_{22}(x_2(t)),$$
$$x_3'(t) = -\alpha x_3(t) + \alpha x_1(t). \quad (5)$$

By the hypothesis (H1), it is easy to see that the origin $(0,0,0)$ is an equilibrium point of system (5). Letting $u_1(t) = x_1(t-\tau_1)$,