Neurocomputing 162 (2015) 85–95

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Analysis of multiple quasi-periodic orbits in recurrent neural networks

R.L. Marichal*, J.D. Piñeiro

Department of Computer and Systems Sciences, University of La Laguna, Avda. Francisco Sánchez S/N, Edf. Informática, 38206 Tenerife, Canary Islands, Spain

ARTICLE INFO

Article history:
Received 30 November 2014
Received in revised form
27 February 2015
Accepted 1 April 2015
Communicated by Rongrong Ji
Available online 17 April 2015

Keywords:
Recurrent neural networks
Bifurcation
Strong resonance
Quasi-periodic orbit

ABSTRACT

In this paper we consider a recurrent neural network model consisting of two neurons and analyze its stability using the associated characteristic model. In order to analyze the multiple quasi-periodic orbits, the strong resonance of this system, in particular that known as the R_2 bifurcation, is also studied. In the case of two neurons, one necessary condition that yields the bifurcation is found. In addition, the direction of the R_2 bifurcation is determined by applying normal form theory and the center manifold theorem. The simple conditions for ensuring the existence of multiple quasi-periodic orbits are given. The strong resonance phenomenon is analyzed using numerical simulations and is related with the codimension-two bifurcation of the high-iteration map.

1. Introduction

The purpose of this paper is to present certain results on the analysis of the dynamics of a recurrent neural network. The particular network in which we are interested is the Williams–Zipser network, also known as a discrete recurrent neural network in [1]. Its two-neuron state equation is

$$x_1(k+1) = f(w_{11}x_1(k) + w_{12}x_2(k)) \quad (1a)$$

$$x_2(k+1) = f(w_{21}x_1(k) + w_{22}x_2(k)), \quad (1b)$$

where $x_i(k)$ is the ith neuron output, w_{ij} are the weight factors of the neuron outputs, and $f(\cdot)$ is a continuous, bounded, monotonically increasing function, such as the hyperbolic tangent.

From the point of view of dynamic system theory, it is interesting to study the equilibrium or fixed points. The dynamics at these points do not change in time. Their character or stability determines the local behavior of nearby trajectories. A fixed point systems can attract (sink), repel (source) or have directions of attraction and repulsion (saddle) of close trajectories [2, Chapter 3]. Besides fixed points, there exist periodic trajectories, quasi-periodic trajectories or even chaotic sets, each with its own stability characterization. All of these features are similar in a class of topologically equivalent systems [3, Chapter 2]. With respect to recurrent neural networks as systems, several dynamics-related results are available in the literature. The most general result is derived in Marcus and Westervelt [4] using the Lyapunov stability theorem. They establish that the only stable equilibrium states that can exist for a symmetric weight matrix are either fixed points or period-two cycles. More recently, Cao [5] proposed less restrictive but more complex conditions. Wang [6] describes an interesting type of trajectory, the quasi-periodic orbits. Passeman [7] obtains some experimental results, such as the coexistence of periodic cycles, chaotic attractors and quasi-periodic trajectories. In [8], Tino gives the position, number and stability types of fixed points for a two-neuron discrete recurrent network with non-zero weights.

The rest of this paper is divided into four additional sections. Section 2 consists of an introduction to bifurcation theory. In Section 3, the local stability of the recurrent neural network and the necessary conditions for the onset of the R_2 strong resonance bifurcation are analyzed. In Section 4, conditions for the direction of the bifurcation and for the existence of quasi-periodic orbits are established. In Section 5, we show the bifurcation diagram and dynamic behavior simulations of the network with the hyperbolic tangent as the activation function.

2. Bifurcation theory overview

In general, when system parameters are slowly changed, the system dynamics varies smoothly. Those dynamics belong to the same class of topologically equivalent systems. Sometimes, the variation of parameters can reach a critical point at which it is no longer topologically equivalent. This is called a bifurcation point [9, Chapter 3], and the system will exhibit new behaviors. In order to determine the new dynamics associated to the bifurcation point, normal form theory is a useful tool [9, Chapter 2]. This theory is a

* Corresponding author. Tel.: +34 922845234; fax: +34 922318288.
E-mail address: rlmarpla@ull.es (R.L. Marichal).

http://dx.doi.org/10.1016/j.neucom.2015.04.001
0925-2312/

technique for transforming the nonlinear dynamical system in critical situation into certain simple standard forms known as normal forms of the bifurcation. In fact, the dynamical behavior of these forms is known and the dynamical system is locally topologically equivalent to them on the bifurcation point. The simplest bifurcations are those associated with the fixed points of the system under analysis. These bifurcations occur at certain critical eigenvalues of the linearized equivalent system at the fixed point. If the bifurcation is characterized by a one-dimensional manifold in parameter space is called a codimension-one bifurcation. More generally, the codimension of a bifurcation is the number of parameters which must be varied for the bifurcation to occur. The three simplest codimension-one bifurcations in discrete-time systems are known as Neimark–Sacker, fold and period-doubling which appear when one eigenvalue is -1, $+1$ and a pair of complex conjugate eigenvalues have a unit modulus, respectively [9, Chapter 4]. The Neimark–Sacker bifurcation is related with an interesting dynamical behavior characterized by the presence of one quasi-periodic orbit. The periodicity of this particular trajectory is irrational, that makes that the set of points it traverses looks like a continuous closed orbit. In the bifurcation curve exist some points where the periodicities of trajectories are rational and appears as a new codimension-two bifurcation. The nomenclature of this rational periodicity is represented by $q:p$, this number indicates that in p mapping iterations, the state completes q revolutions. This terminology is related to the phase locking phenomenon [9, Chapter 7] (also called entrainment or synchronization). Phase locking produces a periodic solution that persists generically as parameters are varied. In contrast, quasi-periodicity is a codimension-one phenomenon, which is thus generically destroyed by perturbation. The result is a well-known bifurcation diagram in the two-parameter plane called the “Arnold tongue” scenario [9, Chapter 7]. The Arnold tongues are situated next to the unit circle, where the rotational number [2] is constant and rational and has a value of p/q, and is surrounded by zones in which it is irrational. For example, in an integrate-and-fire continuous-time neural network model with sinusoidal input, the rational $q:p$ mode-locked solution is identified by a spike train in which p firing events occur in period qf, where f is the forcing input period. In general, codimension-two bifurcations are related with two critical eigenvalues. For example, the rotational number mentioned above is rational then appears a codimension-two bifurcation with the following condition with respect to the eigenvalues:

$$\lambda_{1,2} = e^{\pm i\theta_0}, \quad \theta_0 = \frac{2\pi p}{q}.$$

These bifurcations are known as resonances or strong resonances and they are related with the destruction of quasi-periodic orbits associated to Neimark–Sacker bifurcations. In particular, the R_2 bifurcation is a resonance represented by the rotational number 1:2. In fact, for neural network (1), the numerical simulations on Arnold tongues [10] conclude that the most important Arnold tongue is associated with a 1:2 rotational number, that is, the most frequent resonance is 1:2. Additionally, in R_2 resonance the most interesting dynamical behavior is the presence of multiple quasi-periodic orbits under some conditions of normal form coefficients (see Section 4). With respect to the analysis of bifurcations in neural networks there exist some previous studies. Refs. [11,12] study the discrete-time Hopfield neural network and specifically, their Neimark–Sacker bifurcation and the stability of the associated quasi-periodic orbit. In [10], a simple stability condition for the Neimark–Sacker bifurcation in a two-neuron discrete recurrent neural network is given and the Arnold tongues (related to the phase-locking phenomenon) are studied. In [13], numerical estimates of the Neimark–Sacker bifurcation direction in a Hopfield neural network with two neurons and one time delay are given. Refs. [14,15] complete the bifurcation results for two-neuron discrete-time Hopfield neural networks with time delay and only self-connections between the neurons (no interactions between them). In [16,13,17], results are generalized to n-neuron Hopfield neural networks. In [18], the discrete-time dynamics of a two-neuron network with recurrent connectivity, known as ring neural networks, are studied, showing for specific parameter configurations the relationship between dynamics and the evolution of the external outputs. In [19], the authors consider a system of delayed differential equations representing a simple model for a ring of neurons with some restrictions on the parameters, giving the geometric locus in parameter space that results in a Hopf bifurcation. In [20], a discrete neural network with two neurons is considered and the period-doubling bifurcation is analyzed. In this paper the stability of the bifurcation focuses on the zero fixed point. In contrast, [21] studies the saddle-node, pitchfork and Hopf bifurcations in a recurrent neural network. Additionally, Guo [16] presents some results for a codimension-two bifurcation ring neural network. Finally, Folias and Ermentrout [22] analyze the strong resonance $(1:2)$ of a biological neural network model. With regard to these references, this paper introduces the study of period-doubling at fixed points different from zero, and also includes novel results for strong resonance in high-iteration maps of Hopfield discrete neural networks. Generally, the typical procedure is to analyze the quasi-periodic orbit associated with the Neimark–Sacker bifurcation [16,13,17,10], or to propose conditions for the non-existence of said quasi-periodic orbits by ensuring that the system has fixed stable points [5]. The main novelty of this paper with respect to previous studies is the analysis of multiple quasi-periodic orbits and the destruction process of a quasi-periodic from a numerical simulation approach. Additionally, we show the relationships with the codimension-two bifurcation of the high-iteration map in R_2 strong resonance.

3. Local stability and resonance bifurcation conditions

In the exposition below, a two-neuron neural network is considered. It is usual for the activation function to be a sigmoid function or a tangent hyperbolic function. Here we only need the following assumption:

$$f \in C^1(\Re), \quad f(0) = 0, \quad f'(0) \neq 0, \tag{H.1}$$

where $C^1(\Re)$ is the functions set with continuous first derivative.

In order to simplify the notation we denote (x_1, x_2) as (x, y).

First, the analytical condition of a fixed point can be shown as

$$x = f(w_{11}x + w_{12}y) \tag{2a}$$

$$y = f(w_{21}x + w_{22}y). \tag{2b}$$

Taking into consideration assumption (H.1), it is clear that (0,0) is a fixed point.

Introducing the new variables σ_1 and σ_2, which depend on the diagonal weights and the weight matrix determinant, we have

$$\sigma_1 = \frac{w_{11}f'(f^{-1}(0)) + w_{22}f'(f^{-1}(0))}{2} \tag{3}$$

$$\sigma_2 = |W| f'(f^{-1}(0))^2. \tag{4}$$

The Jacobian matrix of the linearized system evaluated at the fixed point is

$$A = \begin{bmatrix} w_{11}f'(f^{-1}(0)) & w_{12}f'(f^{-1}(0)) \\ w_{21}f'(f^{-1}(0)) & w_{22}f'(f^{-1}(0)) \end{bmatrix} \tag{5}$$