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The resistance distances of electrical networks based on Laplacian generalized inverse

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ABSTRACT

In this paper, the generalized inverse representations for the Laplacian block matrices of graphs $G_1 \boxdot G_2$ and $G_1 \boxminus G_2$ are proposed, based on which the explicit resistance distance can be obtained for the arbitrary two-vertex resistance in the electrical networks. Moreover, some numerical examples are presented, which show the correction and efficiency of the obtained results.

1. Introduction

All graphs considered in this paper are simple and undirected. Let $G=(V(G),E(G))$ be a graph with vertex set $V(E)=\{v_1,v_2,\dots,v_n\}$ and edge set $E(G)=\{e_1,e_2,\dots,e_m\}$. The adjacency matrix of G, denoted by $A(G)$, is the $n\times n$ matrix whose (i,j)-entry is 1 if v_i and v_j are adjacent in G and 0 otherwise. Denote $D(G)$ to be the diagonal matrix with diagonal entries $d_G(v_1),d_G(v_2),\dots,d_G(v_n)$. The Laplacian matrix of G is defined as $L(G)=D(G)-A(G)$. Let $B(G)$ denote the vertex-edge incidence matrix of G, which is the $n\times m$ matrix whose (i,j)-entry is 1 if v_i is incident to e_j and 0 otherwise. The adjacency matrix and the Laplacian matrix of a graph find use in various aspects of structural analysis, and the spectrums of these matrices determine significant topological characteristics of graphs, such as energy, clustering and the number of spanning trees [1,2]. For other undefined notations and terminology from graph theory, the readers may refer to [1] and the references therein.

Klein and Randić [3] introduced a new distance function named resistance distance based on electrical network theory, and they viewed G as an electrical network and replaced each edge e of G with a unit resistance, the resistance distance between any two vertices i and j, denoted by $r_{ij}(G)$, is defined to be the effective resistance between i and j as computed by Ohm's and Kirchhoff's laws [3]. The resistance distances attracted extensive attention of physicists, chemists, as well as mathematicians, due to its wide applications. [4–7,11]. For more information on resistance distances of graphs, the readers are referred to the most recent papers [9,10,12–18,26,28–30].

A large amount of graph operations are introduced in [19–22], such as the Cartesian product, the corona and the edge corona. All these operations are important constructions of different classes networks. It is well known that the subdivision graph $S(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. Let G_1 and G_2 be two vertex disjoint graphs. The following two definitions come from [21].

Definition 1.1 (*See [21]*). The subdivision-vertex neighborhood corona of G_1 and G_2, denoted by $G_1 \boxdot G_2$, is the graph obtained from $S(G_1)$ and $|V(G_1)|$ copies of G_2, all vertex-disjoint, and joining the neighbors of the i-th vertex of $V(G_1)$ to every vertex in the i-th copy of G_2.

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Definition 1.2 (*See [21]*). The subdivision-edge neighborhood corona of G_1 and G_2, denoted by $G_1 \boxminus G_2$, is the graph obtained from $S(G_1)$ and $|I(G_1)|$ copies of G_2, all vertex-disjoint, and joining the neighbors of the i-th vertex of $I(G_1)$ to every vertex in the i-th copy of G_2, where $I(G_1)$ is the set of inserted vertices of $S(G_1)$.

Bu et al. obtained resistance distances in subdivision-vertex join and subdivision-edge join of graphs [7]. Motivated by the above results, in this paper we further investigated the resistance distances in $G_1 \boxdot G_2$ and $G_1 \boxminus G_2$. Comparison to subdivision-vertex join and subdivision-edge join of graphs, the subdivision-vertex neighborhood corona and subdivision-edge neighborhood corona graphs have more vertices and edges, it is clear that handling the problems of the generalized inverse representation for Laplacian matrices and the resistance distances is not an easy task but we deduced it in spectral graph theory approach, which is based on Laplacian generalized inverse.

The rest of the paper is organized as follows. In Section 2, we provide some lemmas and preliminaries. Main results and application are proposed in Sections 3 and 4, respectively. We conclude the paper in Section 5.

2. Preliminaries

The Kronecker product $A \otimes B$ of two matrices $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{p\times q}$ is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. This is an associative operation with the property that $(A \otimes B)^T = A^T \otimes B^T$, and $(A \otimes B)(C \otimes D) = AC \otimes BD$ whenever the products AC and BD exist. The reader is referred to [25] for other properties of the Kronecker product not mentioned here.

The Moore–Penrose pseudo-inverse of matrices have numerous applications in singular differential equations, Markov chains and iterative methods, etc. [27]. We review the definition of $A^{\{1\}}$ inverse of a given matrix A. Let A be a matrix, X is called the $\{1\}$ inverse of A and denoted by $A^{\{1\}}$, if X satisfies the following condition: $AXA=A$. [23] Given a square matrix A, the group inverse of A, denoted by $A^{\#}$, is the unique matrix Xthat satisfies matrix equations $AXA=A, XAX=X, AX=XA$. [7] If A is real symmetric, then $A^{\#}$ exists and $A^{\#}$ is a symmetric $\{1\}-$inverse of A. In fact, $A^{\#}$ is equal to the Moore–Penrose inverse of A since A is symmetric [7].

The Laplacian matrices are singular and the group inverse is exploited in electric network with undirected graphs. For more information, the readers is referred to [7,24]. It is known that resistance distances in a connected graph G can be obtained from any $\{1\}-$inverse of $L(G)$ by the following lemma [7].

Lemma 2.1 (*See [7]*). *Let G be a connected graph, and $(L_G^{(1)})_{ij}$ denote the $(i,j)-$entry of $L_G^{\{1\}}$. Then*

$$r_{ij}(G) = (L_G^{(1)})_{ii} + (L_G^{(1)})_{jj} - (L_G^{(1)})_{ij} - (L_G^{(1)})_{ji} = (L_G^{\#})_{ii} + (L_G^{\#})_{jj} - 2(L_G^{\#})_{ij}.$$

Lemma 2.2 (*See [7]*). *Let $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a nonsingular matrix. If A and D are nonsingular. Then*

$$M^{-1} = \begin{bmatrix} A^{-1}+A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix},$$

where $S = D - CA^{-1}B$ is the Schur complement of A in M.

The following result is very similar to Lemma 2.2, which is proved in [8], thus it is omitted here.

Lemma 2.3 (*See [8]*). *Let $L = \left[\begin{smallmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{smallmatrix}\right]$ be the Laplacian matrix of a connected graph. If L_1 is nonsingular, then*

$$X = \begin{bmatrix} L_1^{-1}+L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} & -L_1^{-1}L_2S^{\#} \\ -S^{\#}L_2^TL_1^{-1} & S^{\#} \end{bmatrix}$$

is a symmetric $\{1\}-$inverse of L, where $S = L_3 - L_2^TL_1^{-1}L_2$.

3. Main results

3.1. Laplacian generalized inverse of subdivision-vertex neighborhood corona graphs $G_1 \boxdot G_2$

We first give the Laplacian generalized inverse of graph $G_1 \boxdot G_2$.

Theorem 3.1. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices, then the following matrix:*

$$\begin{bmatrix} T_1+T_1HS_2^{\#}H^TT_1 & KB^TS_1^{-1}+T_1HS_2^{\#}H^TKB^TS_1^{-1} & T_1HS_2^{\#} \\ KS_1^{-1}B+KS_1^{-1}BHS_2^{\#}H^TT_1 & S_1^{-1}+KS_1^{-1}BHS_2^{\#}H^TKB^TS_1^{-1} & KS_1^{-1}BHS_2^{\#} \\ S_2^{\#}H^TT_1^T & KS_2^{\#}H^TB^TS_1^{-1} & S_2^{\#} \end{bmatrix}$$

is a symmetric $\{1\}-$inverse of $L_{(G_1 \boxdot G_2)}$, where $T_1 = (2+2n_2)^{-1}I_{m_1} + (2+2n_2)^{-2}B^TS_1^{-1}B$, $S_1 = r_1I_{n_1} - (2+2n_2)^{-1}BB^T$, $H = \mathbf{1}_{n_2}^T \otimes B^T$, $K = (2+2n_2)^{-1}$, $S_2 = (L(G_2)+r_1I_{n_2}) \otimes I_{n_1} - [\mathbf{1}_{n_2} \otimes B]T_1[\mathbf{1}_{n_2}^T \otimes B^T]$.

Proof. Let G_1 be an arbitrary r_1-regular graphs with n_1 vertices and m_1 edges, and G_2 be an arbitrary graphs with n_2 vertices. Labelling the vertices of $G_1 \boxdot G_2$ as follows. Let $I(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$, $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(G_2) = \{w_1, w_2, \ldots, w_{n_2}\}$. For $i = 1, 2, \ldots, n_1$, let $w_1^i, w_2^i \ldots, w_{n_2}^i$ denote the vertices of the i-th copy of G_2, with the understanding that w_j^i is the copy of w_j for each j. Denote $W_j = \{w_j^1, w_j^2, \ldots, w_j^{n_1}\}$ for $j = 1, 2, \ldots, n_2$. Then

$$I(G_1) \bigcup V(G_1) \bigcup [W_1 \bigcup W_2 \bigcup \cdots \bigcup W_{n_2}] \tag{1}$$