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Further results on robustness analysis of global exponential stability of recurrent neural networks with time delays and random disturbances

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ABSTRACT

In this paper, further results on robustness analysis of global exponential stability of recurrent neural networks (RNNs) subjected to time delays and random disturbances are provided. Novel exponential stability criteria for the RNNs are derived, and upper bounds of the time delay and noise intensity are characterized by solving transcendental equations containing adjustable parameters. Through the selection of the adjustable parameters, the upper bounds are improved. It shows that our results generalize and improve the corresponding results of recent works. In addition, some numerical examples are given to show the effectiveness of the results we obtained.

1. Introduction

Recurrent neural networks RNNs are nonlinear dynamic systems with some resemblance of biological neural networks in the brain, which include the well-known Hopfield neural networks, cellular neural networks as special cases. In recent decades, many RNNs have been developed and extensively applied to many areas, such as associative memories, image processing, pattern recognition, classification and prediction, signal processing, robotics and control.

The stability of recurrent neural network is necessary for most successful neural network applications. The stability of RNNs depends mainly on their parametrical configuration. In biological neural systems, signal transmission via synapses is usually a noisy process influenced by random fluctuations from the release of neurotransmitters and other disturbances (Haykin, 1994). Moreover, in the implementation of RNNs, external random disturbances and time delays of signal transmission are common and hardly been avoided. It is known that random disturbances and time delays in the neuron activations may result in oscillation or instability of RNNs (Gopalsamy & Leung, 1996; Pham, Pakdaman, & Virbert, 1998). The stability analysis of delayed RNNs (DRNNs) and stochastic RNNs (SRNNs) with external random disturbances have been widely investigated in recent years (see, e.g., Arik, 2002, 2004, Cao, Yuan, & Li, 2006, Chua & Yang, 1988, Faydasicok & Arik, 2013, He, Wu, & She, 2006, Hu, Gao, & Zheng, 2008, Liao, Chen, & Sanchez, 2002, Liao & Wang, 2003, Liu, Wang, & Liu, 2006, Wang, Liu, Li, & Liu, 2006, Xu, Lam, & Ho, 2006, Yuan, Cao, & Li, 2006, Zeng & Wang, 2006a, 2006b, Zhang & Jin, 2000, Zhang, Wang, & Liu, 2009 and Zhu, Shen, & Chen, 2010a, 2010b and the references cited therein).

It is well known that noise and time delays can lead to instability and they can destabilize stable RNNs if they exceed their limits. The instability depends on the intensity of the noise and time delays (Mao, 2007). For a stable RNN, if the noise intensity is low and time delay is small, the perturbed RNN may still be stable. Therefore, it is interesting to determine how much time delay and random disturbance that a stable RNN can withstand without losing its global exponential stability. Although various stability properties of RNNs have been analyzed extensively in recent years by using the Lyapunov and the linear matrix inequality (LMI) methods (Chen & Lu, 2008; Huang, Ho, & Lam, 2005; Zhang & Wang, 2008; Zheng, Shan, Zhang, & Wang, 2013), the robustness of the global stability of RNNs is rarely investigated directly by estimating the upper bounds of noise level and time delays.

Motivated by the above discussions, our purpose is to quantify the parameter uncertainty level for stable RNNs in this paper. Compared with the conventional Lyapunov stability theory and linear matrix inequality methods, we investigate the robust stability for global exponential stability directly from the coefficients of the

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RNNs which should satisfied the global exponential stability condition. In this paper, we further characterize the robustness of RNNs with time delays and additive noise by deriving the upper bounds of delays and noise for global exponential stability. Novel exponential stability criteria for the RNNs are derived, and upper bounds of the time delay and noise intensity are estimated by solving transcendental equations containing adjustable parameters. We generalize and improve previous results by choosing adjustable parameters. Moreover, we prove theoretically that, for any globally exponentially stable RNNs, if additive noise and time delays are smaller than the derived upper bounds herein, then the perturbed RNNs are guaranteed to be globally exponentially stable.

2. Problem formulation

Throughout this paper, unless otherwise specified, R^n and $R^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of $n \times m$ real matrices. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). $\omega(t)$ be a scalar Brownian motion defined on the probability space. If A is a matrix, its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$, where $|\cdot|$ is the Euclidean norm. Denote $L^2_{\mathscr{F}_0}([-\bar{\tau}, 0]; R^n)$ as the family of all $\mathscr{F}_0-$ measurable $C([-\bar{\tau}, 0]; R^n)$ valued random variables $\psi = \{\psi(\theta) : -\bar{\tau} \leq \theta \leq 0\}$ such that $\sup_{-\bar{\tau}\leq\theta\leq 0} E|\psi(\theta)|^2 < \infty$ where $E\{\}$ stands for the mathematical expectation operator with respect to the given probability measure P.

Consider an RNN model

$$\dot{z}(t) = -Az(t) + Bg(z(t)) + u,$$
$$z(t_0) = z_0, \tag{1}$$

where $z(t) = (z_1(t), \ldots, z_n(t))^T \in R^n$ is the state vector of the neurons, $t_0 \in R_+$ and $z_0 \in R^n$ are the initial values, $A = \mathrm{diag}\{a_1, \ldots, a_n\} \in R^{n\times n}$ is the self-feedback connection weight matrix, $B = (b_{kl})_{n\times n} \in R^{n\times n}$ is the connection weight matrix, u is the neuron external input (bias), $g(z(t)) = (g_1(z_1(t)), \ldots, g_n(z_n(t)))^T \in R^n$ is a vector-valued activation function which satisfies the global Lipschitz condition, i.e.,

$$|g(u) - g(v)| \leq k|u - v|, \quad \forall u, v \in R^n, \tag{2}$$

where k is a known constant.

In addition, we assume that RNN (1) has an equilibrium point $z^* = (z_1^*, z_2^*, \ldots, z_n^*)^T \in R^n$. Let $x(t) = z(t) - z^*, f(x(t)) = g(x(t) + z^*) - g(z^*)$, and then, (1) can be rewritten as

$$\dot{x}(t) = -Ax(t) + Bf(x(t)),$$
$$x(t_0) = x_0, \tag{3}$$

where $x_0 = z_0 - z^*$ i.e., the origin is an equilibrium point of (3). Hence, the stability of the equilibrium point z^* of (1) equals to the stability of origin point in the state space of (3). In addition, the function f in (3) satisfies the following Lipschitz condition and $f(0) = 0$.

Assumption 1. The activation function $f(\cdot)$ satisfies the following Lipschitz condition; i.e.,

$$|f(u) - f(v)| \leq k|u - v|, \quad \forall u, v \in R^n, \ f(0) = 0, \tag{4}$$

where k is a known constant.

It is known that, based on Assumption 1, RNN (3) has a unique state $x(t; t_0, x_0)$ on $t \geq t_0$ for any initial value t_0, x_0. The origin is the equilibrium point of RNN (3), as $f(0) = 0$. Now we define the global exponential stability of the state of RNN (3).

Definition 1. The state of RNN (3) is globally exponentially stable, if for any t_0, x_0, there exist $\alpha > 0$ and $\beta > 0$ such that

$$|x(t; t_0, x_0)| \leq \alpha|x(t_0)| \exp(-\beta(t - t_0)), \quad \forall t \geq t_0, \tag{5}$$

where $x(t; t_0, x_0)$ is the state of the model in (3).

Numerous criteria for ascertaining the global exponential stability of RNN (3) have been developed, e.g., Chen (2001), Shen and Wang (2007, 2008, 2012), Zeng, Wang, and Liao (2005), Zhu and Shen (2013) and Zhu et al. (2010a, 2010b).

3. Main results

In the following equation, we consider the noise-induced stochastic RNNs (SRNNs) described by the Itô stochastic equation

$$dy(t) = [-Ay(t) + Bf(y(t))]dt + \sigma y(t)d\omega(t),$$
$$y(t_0) = x_0, \tag{6}$$

with the initial data $y(t_0) = x_0 \in R^n$, and A, B, f, in here are the same in (3), σ is the intensity of noise, $\omega(t)$ is a scalar Brownian motion defined on the probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$. Under Assumption 1, SRNN (6) has a unique state $y(t; t_0, x_0)$ on $t \geq t_0$ for any initial value t_0, x_0; the origin point $y = 0$ is the equilibrium point. Now, the question is, given globally exponentially stable RNN (3), how much the noise intensity will not derail the stability of RNNs? We will characterize how much the stochastic perturbation can bear such that SRNN (6) continues to remain globally exponentially stable. For the SRNN model (6), we give the following definition of global exponential stability.

Definition 2 (*Mao, 2007*)**.** SRNN (6) is said to be almost surely globally exponentially stable if for any $t_0 \in R_+, x_0 \in R^n$, there exist $\alpha > 0$ and $\beta > 0$ such that $\forall t \geq t_0, |y(t; t_0, x_0)| \leq \alpha|y(t_0)| \exp(-\beta(t - t_0))$ holds almost surely, i.e., the Lyapunov exponent $\limsup_{t\to\infty}(\ln |y(t; t_0, x_0)|/t) < 0$ almost surely, where $y(t; t_0, x_0)$ is the state of SRNN (6). SRNN (6) is said to be mean square globally exponentially stable if for any $t_0 \in R_+$, $x_0 \in R^n$, there exist $\alpha > 0$ and $\beta > 0$ such that $\forall t \geq t_0$, $E|y(t; t_0, x_0)|^2 \leq \alpha|y(t_0)| \exp(-\beta(t - t_0))$ holds; i.e., the Lyapunov exponent $\limsup_{t\to\infty}(\ln(E|y(t; t_0, x_0)|^2)/t) < 0$, where $y(t; t_0, x_0)$ is the state of SRNN (6).

From the definition, in general, almost surely global exponential stability and mean square one do not imply each other (Mao, 2007). However, if Assumption 1 holds, we have the following lemma (Mao, 2007, Theorem 4.2, p. 128).

Lemma 1. *Let Assumption 1 hold. The global exponential stability in sense of mean square of SRNN (6) implies the almost surely global exponential stability of SRNN (6).*

Theorem 1. *Let Assumption 1 hold and RNN (3) be globally exponentially stable. Noise-induced SRNN (6) is mean square globally exponentially stable and also almost surely globally exponentially stable, if $|\sigma| < \bar{\sigma}$ is a unique positive solution of the transcendental equation*

$$\frac{2\sigma^2\alpha^2}{(1-\varepsilon)\beta} \exp\left\{4\Delta\left[\frac{2\Delta}{\varepsilon}(\|A\|^2 + \|B\|^2k^2) + \frac{\sigma^2}{1-\varepsilon}\right]\right\}$$
$$+ 2\alpha^2 \exp(-2\beta\Delta) = 1, \tag{7}$$

where ε is an adjustable parameter, $\varepsilon \in (0, 1)$ and $\Delta > \ln(2\alpha^2)/(2\beta) > 0$.

Proof. For simplify, we will denote $x(t; t_0, x_0)$ and $y(t; t_0, x_0)$ as $x(t)$ and $y(t)$, respectively. From (3) and (6), we have

$$x(t) - y(t) = \int_{t_0}^{t} [-A(x(s) - y(s)) + B(f(x(s)) - f(y(s)))]ds$$
$$- \int_{t_0}^{t} \sigma y(s)d\omega(s).$$