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Hopf Bifurcation in a Chaotic Associative Memory

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ABSTRACT

This paper has two basic objectives: the first is to investigate Hopf Bifurcation in the internal state of a Chaotic Associative Memory (CAM). For a small network with three neurons, resulting in a six-dimensional Equation of State, the existence and stability of Hopf Bifurcation were verified analytically. The second objective is to study how the Hopf Bifurcation changed the external state (output) of CAM, since this network was trained to associate a dataset of input–output patterns. There were three main differences between this study and others: the bifurcation parameter was not a time delay, but a physical parameter of a CAM; the weights of interconnections between chaotic neurons were neither free parameters nor chosen arbitrarily, but determined in the training process of classical AM; the Hopf Bifurcation occurred in the internal state of CAM, and not in the external state (input–output network signal). We present three examples of Hopf Bifurcation: one neuron with supercritical bifurcation while the other two neurons do not bifurcate; two neurons bifurcating into a subcritical bifurcation and one neuron does not bifurcate; and the same example as before, but with a supercritical bifurcation. We show that the presence of a limit cycle in the internal state of CAM prevents output signals from the network converging towards a desired equilibrium state (desired memory), although the CAM is able to access this memory.

1. Introduction

Since Hopfield proposed the Recurrent Neural Network (RNN) [1], a neural network with nonlinear behavior in which the output signal is reintroduced into the network, RNN has been used to model nonlinear dynamic systems, as an alternative to the traditional Ordinary Differential Equation approach.

Such Nonlinear Systems (NLS) can present different dynamical behaviors, such as instability, asymptotic stability, periodicity and chaos [2]. The study of the changes of dynamical behaviors of NLS as a function of variation of the parameter values includes the study of Bifurcation [3]. There are studies in which local bifurcations were verified in general RNNs [4–7], in Associative Memories (AM) and in Bidirectional Associative Memories (BAM) [5,8–20].

For neural networks, the typical goals were to prove the existence and stability of Hopf Bifurcation for continuous systems [9,12,15], and for discrete systems [4,5,10,16,18], deemed Neimark–Sacker Bifurcation [3]. More recently, several studies with Neural Networks in which the bifurcation parameter $\mu \in \mathbb{R}^2$ have been presented and show the occurrence of other types of bifurcations such as Hopf–Hopf bifurcation, when there are two pairs of purely complex eigenvalues [20]; Zero-Hopf Bifurcation, when there is the presence of a zero eigenvalue and a pair of purely complex eigenvalues [21]; Bautin Bifurcation, when there is a pair of purely imaginary eigenvalues and the first Lyapunov coefficient for the Hopf Bifurcation when it converges to zero [14]; or Bogdanov–Takens Bifurcation, when a null eigenvalue has a multiplicity of two [6,22,23].

A time delay is usually used as a bifurcation parameter for Associative Memory analysis. Such a parameter was taken in different versions such as the number of input time delays in a neuron, the sum of input time delays of all network neurons, or the average of an input time delay interval. The network weights were either assumed to be free parameters or were set up arbitrarily to establish bifurcation conditions.

There are few recent studies dealing with Hopf Bifurcations in AM with chaotic neurons [24–26]. Bifurcation studies on AM and BAM with the chaotic neuron of Adachi and Aihara [27] are very often focused on period-doubling bifurcation and chaos [28–30]. In [24], the treatment of Center Manifold Reduction and Normal Forms for a continuum 3-dimensional CAM was applied. In [25,26], the authors presented only numerical results for a Discrete BAM with chaotic neurons. In none of the three papers were the weights of the Neural Network calculated as the classical neural training process does [27].

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In an earlier paper, Araujo et al. [29] presented a chaotic BAM (CBAM) as a generalization of the chaotic neuron [27] proposed for Autoassociative Memories. Different dynamic behaviors such as double-period bifurcation, deterministic chaos and crises were verified numerically for CBAM. In [31], the chaos in CBAM was controlled with a pinning control method and was used to show how to skip intermediate steps in a task that consisted of a sequence of steps [32]. In this case, a chaotic neuron had its dynamics determined by four parameters, but neither represented time delay explicitly.

This paper has two basic objectives: the first is the analytical and numerical study of the existence and stability of Hopf Bifurcation in the internal state of a CAM [27], considering a single bifurcation parameter. The second objective is to study how the Hopf Bifurcation changed the external state (output) of the CAM, since this network was trained to autoassociate a dataset of patterns. The main differences between the current study and previous studies are:

- Analytical studies of Hopf Bifurcation occur in low-dimensional systems. The largest dimension in previous studies [18] we find is six, the same dimensions as our case.
- While [27–29,31] worked with chaotic neural networks and focused on the study of Double-period Bifurcation, chaos and its control, our work is focused on the study of Hopf Bifurcation.
- The papers that studied the Hopf Bifurcation in chaotic neural networks either executed analytical analysis of Hopf Bifurcation for a smaller dimension (3) [24] or studied Hopf bifurcation numerically [25,26]. We studied analytical and numerical analysis for a six dimensional system.
- All these studies [5,9–20] related the bifurcation parameter to time delay, while in our case the bifurcation parameter is a damping parameter of the system. Besides, in these studies the Hopf Bifurcation occurs in the input/output space of phase, while in our case this occurs in the internal state space of phases.
- The interconnection weights w_{ij} in our case were determined by a training process to recall a dataset of training memories, thus establishing a relation between the input and the output signal. In other papers, the weights w_{ij} usually are free parameters whose values are chosen to establish the conditions for local stability of the equilibrium point, or w_{ij} and a_i were set arbitrarily.

The next section presents a general CAM. Section 3 introduces the methodology chosen, following the same steps as by Crawford [33]. First, Center Manifold Reduction is presented, and then Hopf Bifurcation is described in Normal Forms. The analytical deduction of the existence and stability of a Hopf Bifurcation, in a small CAM with three neurons, is presented in Section 4. Numerical simulations for this system are presented in Section 5. The conclusions and final considerations are in Section 6.

2. Chaotic Autoassociative Memory

Chaotic Associative Memory [27] is defined as a neural network trained with a dataset of p input binary or bipolar pattern vectors, $\{\mathbf{y}_i^d\}_{i=1}^p$ with a dimension of n, $\mathbf{y}^d \in \mathbb{R}^n$. The output of each neuron of the network is fed back into all neurons of the network. The chaotic neuron has two internal states η_i and ζ_i, which are free variables of an activation function f that produces an output $y_i(k+1)$, as shown below for the i-th network neuron:

$$\begin{cases} y_i(k+1) = f_i(\eta_i(k+1)+\zeta_i(k+1)) \\ \eta_i(k+1) = k_f \eta_i(t) + \sum_{j=1}^{n} w_{ij} y_j(k) \\ \zeta_i(k+1) = k_r \zeta_i(k) - \rho y_i(k) + a_i \end{cases} \tag{1}$$

where $i = 1,..,n$; k_f and k_r are, respectively, the decay parameters for feedback inputs and the decay parameter of refractoriness; ρ is the scaling refractory parameter, and $a_i = a$ is an external stimulation, assumed to be constant without loss of generality. These four parameters can be tuned to generate distinct dynamics. The activation function f has the form of a *logistic function* $f(u) = 1/(1+e^{-\beta u})$. The parameter β is usually assumed to have a high value (~ 50) and the logistic function behavior is approximately a step function. When the chaotic parameters are turned off, $k_f = k_r = \rho = a = 0$, the chaotic autoassociative memory (1) enters the classical autoassociative mode, depending only on the weighted input signal of all neurons.

The network process of learning the dataset of samples entails calculating the interconnection weight matrix $\mathbf{W}$ as

$$\mathbf{W} = \frac{1}{p}\sum_{i=1}^{p} \mathbf{y}_i^d (\mathbf{y}_i^d)^t \tag{2}$$

where $\mathbf{y}_i^d$ is a bipolar column-vector pattern of a training dataset. If $\mathbf{y}^d$ is originally binary, it should be transformed to bipolar, only to calculate the matrix weight $\mathbf{W}$.

Three adaptations were considered to the original model of CAM described above: (i) the external stimulation parameter a was assumed to be null, $a=0$; (ii) the activation function was assumed to be a *hyperbolic tangent function*, so as to work with bipolar samples

$$f(u) = \frac{e^{\beta u} - e^{-\beta u}}{e^{\beta u} + e^{-\beta u}} \tag{3}$$

and (iii) the parameter β was relaxed so as to assume small positive values. The first two modifications aim to ensure that the state $\mathbf{y} = \mathbf{0}$ is an equilibrium point of (1). The relaxation in the value of β parameter was necessary to numerically observe Hopf Bifurcation.

The CAM has four parameters k_f, k_r, ρ and β that can be adjusted, thereby directly influencing the internal dynamics of the system. For our study, we assumed a simplification of the problem: the CAM will have its internal dynamics governed by a single bifurcation parameter, the refractoriness parameter k_r. Previous studies of the chaotic neuron based on factorial analysis, showed that k_r is the most important parameter for the occurrence of chaos [27,31], so it is considered to be the bifurcation parameter, $\mu = k_r$.

Since the number of chaotic neurons in the CAM is equal to the dimension of input/output patterns, we define a general internal state of CAM $\mathbf{x}$, as $\mathbf{x} = [x_1\ x_2\ \cdots\ x_{2n}]^T = [\zeta_1\ \cdots\ \zeta_n\ \eta_1 \cdots \eta_n]^T$, $\mathbf{x} \in \mathbb{R}^{2n}$. The internal Equation of State of CAM is given by

$$\frac{d}{dk}\mathbf{x}(k) = \dot{\mathbf{x}} \equiv \mathbf{x}(k+1) = \mathbf{C}\mathbf{x}(k) + \mathbf{b}(k) \tag{4}$$

where the components, linear $\mathbf{C}\mathbf{x}(k)$ and nonlinear $\mathbf{b}(k)$, were defined as

$$\mathbf{C}\mathbf{x}(k) = \begin{pmatrix} k_r & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & k_r & 0 & 0 & 0 \\ 0 & 0 & 0 & k_f & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & k_f \end{pmatrix} \begin{pmatrix} x_1(k) \\ \vdots \\ x_n(k) \\ x_{n+1}(k) \\ \vdots \\ x_{2n}(k) \end{pmatrix} \tag{5}$$

and

$$\mathbf{b}(k) = \begin{pmatrix} -\rho f(x_1(k)+x_{n+1}(k)) \\ \vdots \\ -\rho f(x_n(k)+x_{n+n}(k)) \\ \sum_{j=1}^{n} w_{1j} f(x_j(k)+x_{n+j}k)) \\ \vdots \\ \sum_{j=1}^{n} w_{nj} f(x_j(k)+x_{n+j}(k)) \end{pmatrix} \tag{6}$$