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Multistability and complete convergence analysis on high-order neural networks with a class of nonsmooth activation functions

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ABSTRACT

In this paper, we are concerned with a class of high-order neural networks (HONNs) with nonsmooth activation functions. A set of new sufficient conditions ensuring the coexistence of 3^n equilibrium points and the local stability of 2^n equilibrium points are proposed, which reveal that the high-order interactions between neurons also play an important role on the multistability of HONNs. Besides, every solution is shown to converge to a certain equilibrium point, that is, the systems are also completely stable. Furthermore, for the 2-neuron neural networks, we can get that the stable manifolds of unstable equilibrium points constitute the boundaries of attraction basins of stable equilibrium points, despite the nonlinearity of high-order items of HONNs. Several numerical examples are presented to illustrate the effectiveness of our criteria.

1. Introduction

Neural networks have been extensively studied in recent years, due to their wide applications in image processing, pattern recognition, associative memories, combinational optimization and many other fields. As revealed in [1], the neural networks with first-order connections are shown to have limitations such as limited capacity when used in pattern recognition and optimization problems. By introducing the high-order interactions between neurons, the neural networks can be with impressive computational, storage, and learning capabilities [2], and have stronger approximation property, faster convergence rate, greater storage capacity and higher fault tolerance than traditional first-order neural networks [3,13]. Therefore, considerable interests have been attracted to study the high-order neural networks.

Consider the following high-order neural networks described by:

$$\frac{du_i(t)}{dt} = -d_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t)) + \sum_{j=1}^{n}\sum_{l=1}^{n} c_{ijl} f_j(u_j(t)) f_l(u_l(t)) + I_i, \quad i = 1, \dots, n, \tag{1}$$

where $u_i(t)$ represents the state of the ith unit at time t; $d_i > 0$ denotes the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; a_{ij}, c_{ijl} correspond to the first-order and high-order connection weights on the ith unit, respectively; $f_j(\cdot)$ is the activation function; I_i stands for the external input, $i, j, l = 1, \dots, n$.

In [7,8,13,19], the high-order neural networks, either with bounded activation functions or with unbounded activation functions, were investigated, and a set of sufficient conditions guaranteeing the existence, uniqueness of the equilibrium point or almost periodic solution, and its global attractivity were presented.

Recently, it is revealed that the multistable dynamics play an essential role in some neuromorphic analog circuits [5,6], that is, the system exhibits a large number of equilibrium points, and more than one equilibrium points are stable. Also in many applications, such as associative memory, pattern recognition and decision making, it is desirable that the neural networks involved can have multiple attractors. Thus, it is of great importance to study the dynamics of multistable neural networks in both theory and applications.

By decomposing the phase space $\mathbb{R}^n$ into 3^n subset regions, the multistability and the multiperiodicity of neural networks were studied in [9], which showed that the systems can exhibit 2^n locally stable periodic solutions located in 2^n subset regions. In [10,14], it indicated that the neural networks can have 3^n equilibrium points,

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and 2^n of them are locally stable, under some conditions. A class of neural network models with piecewise linear activation functions with $2r$ corner points were also addressed in [16]. By rigorous analysis, it was pointed out that there are $(2r+1)^n$ equilibrium points in all, located in each subset region of $\mathbb{R}^n$, and $(r+1)^n$ of them are locally stable and others are unstable. Besides, a novel approach was proposed to depict the attraction basins of stable equilibrium points for neural networks with two neurons [16] and with n-neurons [17], and the complete stability of systems can also be derived directly. For more references, refer to [15,18,20–27] and so on.

It is worth noting that these works only focused on the first-order neural networks. In [21,22], the authors studied a class of high-order competitive neural networks, and obtained the coexistence of 2^n equilibrium points and their local stability under some conditions. However, it is still unknown whether there are any equilibrium points located in the remaining 3^n-2^n subset regions, and the dynamics of solution trajectories in these regions. To the best of our knowledge, these concerns are still open and have not been properly studied.

In this paper, we are to consider the high-order neural networks (1) with these concerns. Note that the nonsmooth functions are widely employed as activation functions in neural network models and play an important role in designing neural networks [4,11]. Here, we consider a class of continuous activation functions defined as follows:

$$f_j(x)=\begin{cases} m_j & -\infty<x<p_j,\\ \bar{f}_j(x) & p_j\le x\le q_j,\\ M_j & q_j<x<+\infty, \end{cases} \tag{2}$$

where $\bar{f}_j$ is monotonically nondecreasing and with that

$$0\le \gamma_j\le \frac{\bar{f}_j(x)-\bar{f}_j(y)}{x-y}\le \sigma_j<+\infty, \tag{3}$$

for any $x,\ y\in\mathbb{R}$ and $j=1,\dots,n$.

By rigorous analysis, we are to show that there are 3^n equilibrium points of system (1) existed in all under some mild conditions, 2^n of them are local stable and others are unstable. And each solution would converge to some equilibrium point, that is, the system (1) is also completely stable. In addition, we extend the approach proposed in [16] to address the attraction basins of stable equilibrium points for the 2-neuron high-order neural networks. It is shown that, despite the nonlinearity of high-order items in system (1), the combination of stable manifolds of unstable equilibrium points can also split the phase space $\mathbb{R}^2$ into four open connected domains, which are the exact attraction basins of stable equilibrium points. Several illustrative examples are provided to verify the effectiveness of our results.

2. Preliminaries

First of all, we present some definition and denotations that will be used in the following sections.

Definition 1. The high-order neural network (1) is said to be completely stable if for any initial value, the corresponding solution trajectory converges to a certain equilibrium point.

By the geometrical configuration of activation function (2), for each $i=1,\dots,n$, we denote an index $\xi_i\in\{-1,0,1\}$ and use it to define the interval of $\mathbb{R}$ as follows:

$$I_{\xi_i}=\begin{cases} (-\infty,p_i), & \xi_i=-1\\ [p_i,q_i], & \xi_i=0\\ (q_i,+\infty), & \xi_i=1 \end{cases}$$

Therefore, the whole space $\mathbb{R}^n$ can be divided into 3^n subset regions as

$$\Phi_\xi=\prod_{k=1}^{n} I_{\xi_k}$$

with index vectors $\xi=(\xi_1,\dots,\xi_n)\in\{-1,0,1\}^n$, and $\prod$ denoting the Cartesian product from left to right.

For each subset region Φ_ξ, we also define the corresponding index subsets as follows:

$$\mathbb{N}_1^\xi=\{i:\xi_i=-1\},\quad \mathbb{N}_2^\xi=\{i:\xi_i=0\},\quad \mathbb{N}_3^\xi=\{i:\xi_i=1\}. \tag{4}$$

Denote the number of zeros in the vector ξ as $\delta(\xi)=\#\mathbb{N}_2^\xi$.

3. Main results

3.1. Multistability and complete stability of HONNS

In this section, we first investigate the number of equilibrium points of neural networks (1) existed in all, and we have

Lemma 1. *Suppose that*

$$\begin{cases} -d_ip_i+a_{ii}m_i+c_{iii}m_i^2+\sum\limits_{j=1,j\ne i}^{n}\max\{a_{ij}m_j,a_{ij}M_j\}\\ \quad+\sum\limits_{j=1,j\ne i}^{n}\max\{(c_{iij}+c_{iji})m_im_j,(c_{iij}+c_{iji})m_iM_j\}\\ \quad+\sum\limits_{j=1,j\ne i}^{n}\sum\limits_{l=1,l\ne i}^{n}\max\{c_{ijl}m_jm_l,c_{ijl}m_jM_l,c_{ijl}M_jm_l,c_{ijl}M_jM_l\}+I_i<0\\ -d_iq_i+a_{ii}M_i+c_{iii}M_i^2+\sum\limits_{j=1,j\ne i}^{n}\min\{a_{ij}m_j,a_{ij}M_j\}\\ \quad+\sum\limits_{j=1,j\ne i}^{n}\min\{(c_{iij}+c_{iji})M_im_j,(c_{iij}+c_{iji})M_iM_j\}\\ \quad+\sum\limits_{j=1,j\ne i}^{n}\sum\limits_{l=1,l\ne i}^{n}\min\{c_{ijl}m_jm_l,c_{ijl}m_jM_l,c_{ijl}M_jm_l,c_{ijl}M_jM_l\}+I_i>0 \end{cases} \tag{5}$$

holds for all $i=1,\dots,n$. *Then system* (1) *can have* 3^n *equilibrium points.*

Proof. For each subset region Φ_ξ, we will prove that there exists at least one equilibrium point.

In fact, by condition (5), we can pick a small positive constant ϵ such that

$$\begin{cases} -d_i(p_i-\epsilon)+a_{ii}m_i+c_{iii}m_i^2+\sum\limits_{j=1,j\ne i}^{n}\max\{a_{ij}m_j,a_{ij}M_j\}\\ \quad+\sum\limits_{j=1,j\ne i}^{n}\max\{(c_{iij}+c_{iji})m_im_j,(c_{iij}+c_{iji})m_iM_j\}\\ \quad+\sum\limits_{j=1,j\ne i}^{n}\sum\limits_{l=1,l\ne i}^{n}\max\{c_{ijl}m_jm_l,c_{ijl}m_jM_l,c_{ijl}M_jm_l,c_{ijl}M_jM_l\}+I_i<0\\ -d_i(q_i+\epsilon)+a_{ii}M_i+c_{iii}M_i^2+\sum\limits_{j=1,j\ne i}^{n}\min\{a_{ij}m_j,a_{ij}M_j\}\\ \quad+\sum\limits_{j=1,j\ne i}^{n}\min\{(c_{iij}+c_{iji})M_im_j,(c_{iij}+c_{iji})M_iM_j\}\\ \quad+\sum\limits_{j=1,j\ne i}^{n}\sum\limits_{l=1,l\ne i}^{n}\min\{c_{ijl}m_jm_l,c_{ijl}m_jM_l,c_{ijl}M_jm_l,c_{ijl}M_jM_l\}+I_i>0 \end{cases} \tag{6}$$

holds for all $i=1,\dots,n$.

Let $u(t)$ be a solution of system (1) with initial state $u(0)\in\Phi_\xi$. For $i\in\mathbb{N}_1^\xi$, if there exists some $t_0\ge 0$ such that $p_i-\epsilon\le u_i(t_0)<p_i$,