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The existence of periodic solutions for coupled systems on networks with time delays

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ABSTRACT

This paper is concerned with the existence of periodic solutions for coupled systems on networks with time delays (CSND). By the combined method of graph theory, coincidence degree theory and Lyapunov method, a systematic approach for the existence of periodic solutions to CSND is developed. We apply this approach to a coupled system of nonlinear oscillators with time delays and obtain the existence of periodic solutions. Finally, a numerical example is provided to illustrate the effectiveness of the results developed.

1. Introduction

The existence of periodic solutions of ordinary and functional differential equations has been studied extensively due to its universal importance. And coincidence degree theory, the upper and lower solutions method, fixed point theorem, bifurcation theory, and Lyapunov method are often used to prove the existence of periodic solutions. In [1–8], by applying continuation theorem of coincidence degree, a large amount of sufficient conditions have been derived to guarantee the existence of periodic solutions of many kinds of systems.

On the other hand, coupled systems on networks have attracted great attention in biological systems, artificial neural networks, internet and social networks, complex ecosystems, the spread of infectious diseases, and so on [9–14]. Based on graph theory, coupled systems on networks can be described by a directed graph, in which each vertex represents an individual system called vertex system and the directed arcs stand for the inter-connections and interactions among vertex systems [15]. In recent years, the dynamical behaviors of coupled systems on networks have been widely studied, and many good results have been reported. In [12–14,16–18], the synchronization problems of the complex networks have been considered. By applying graph-theoretic approach, papers [15,19–23] studied the global stability for different kinds of coupled systems on networks, and provided systematic methods for constructing the global Lyapunov functions. At the same time, the periodicity of coupled systems has also attracted great interests of scientists, and lots of results have been reported, see [24–32] and the references therein. And in these papers, by constructing suitable Lyapunov functionals, applying coincidence degree theory and some analysis techniques, sufficient conditions for the existence of the periodic solutions to different neural networks are obtained. However, few papers combined with the topology of the network. Therefore, in this paper, by using graph-theoretic approach, the existence of periodic solutions of coupled systems on networks with time delays (CSND) is investigated.

Given a digraph $\mathcal{G}$ with l ($l \geq 2$) vertices, assume that each vertex dynamics is described by differential equations with time delays

$$\dot{x}_k(t) = f_k(x_k(t), x_k(t-\tau_k), t), \quad 1 \leq k \leq l,$$

where $x_k(t) = (x_{k1}(t), x_{k2}(t), \ldots, x_{km}(t))^T$, τ_k stands for the time delay in the k-th subsystem, $f_k = (f_{k1}, f_{k2}, \ldots, f_{km})^T : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m$ is a continuous function, and $f_k(\cdot,\cdot,t) = f_k(\cdot,\cdot,t+T)$ for some $T > 0$, $\mathbb{R}_+ = [0, +\infty)$. If we assume that the influence of the h-th subsystem on the k-th subsystem is described by $g_{kh}(x_k(t-\tau_k), x_h(t-\tau_h), t)$, $g_{kh} = 0$ if and only if there exists no influence from h-th subsystem to k-th subsystem, then we can obtain a kind of CSND below

$$\dot{x}_k(t) = f_k(x_k(t), x_k(t-\tau_k), t) + \sum_{h=1}^{l} g_{kh} \times (x_k(t-\tau_k), x_h(t-\tau_h), t), \quad 1 \leq k, \ h \leq l, \ 0 \leq \tau_k, \ \tau_h < \infty, \tag{1}$$

where $g_{kh} : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m$ is continuous, $g_{kh}(\cdot,\cdot,t) = g_{kh}(\cdot,\cdot,t+T)$. The initial value of system (1) is usually given by

$$x(t) = \phi(t),$$

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where $\phi(t)=(\phi_1(t),\phi_2(t),\dots,\phi_l(t))^T$ is a continuous vector function on $[-\tau,0]$, in which $\tau=\max\{\tau_1,\tau_2,\dots,\tau_l\}$.

The dynamics of system (1) depends on not only the individual vertex dynamics but also the coupling topology, meanwhile, system (1) exhibits a great of complexity, which are mainly caused by the nonlinearity, periodic coefficients and time delays. Hence, studying such system is meaningful and challenging. In this paper, we mainly focus on the existence of periodic solutions to system (1). By employing graph theory, Lyapunov method and coincidence degree theory, a systematic approach for the existence of periodic solutions of system (1) is developed. The methods of estimating the bound of periodic solutions to the auxiliary equation of system (1) are Lyapunov method and graph theory, which are different from that in the above cited papers. We will show that our approach can be applied to coupled oscillators on a network, a numerical example is given to illustrate the correctness of the developed theory.

The contributions and novelties of the current work are as follows:

1. Graph theory, Lyapunov method and coincidence degree theory are combined together to study existence of periodic solutions of coupled systems on networks.
2. A systematic approach for the existence of periodic solutions to general system (1) is obtained.
3. This result is applied to nonlinear coupled oscillators on a network with time delays, and the sufficient conditions for the existence of periodic solutions are derived.

The paper is organized as follows: In the following section, we give some useful lemmas. Our main results are presented in Section 3. Then in Section 4, our results are applied to coupled oscillators on a network to demonstrate their applicability and effectiveness. Finally, a numerical example is given to illustrate the correctness of the developed theory.

2. Main lemmas

The following process is standard and motivated by [5]. Let $X=Z=\{x\in C(\mathbb{R},\mathbb{R}^{ml}):x(t+T)=x(t)\}$ be equipped with the norm $\|x\|=\left(\sum_{k=1}^{l}\max_{t\in[0,T]}(\sum_{i=1}^{m}|x_{ki}(t)|^2)\right)^{1/2}$. Then X and Z are Banach spaces. Set

$$u_k(t)=\left(f_k(x_k(t),x_k(t-\tau_k),t)+\sum_{h=1}^{l}g_{kh}(x_k(t-\tau_k),x_h(t-\tau_h),t)\right)^T.$$

Define an operator L in the following form:

$$L:\mathrm{Dom}\,L\subset X\to X,\quad Lx=x'$$

and

$$N:X\to X,\quad Nx=(u_1,u_2,\dots,u_l)^T.$$

It is not difficult to show that $\mathrm{Ker}\,L=\left\{x\in X:x=c\in\mathbb{R}^{lm}\right\}$, $\mathrm{Im}\,L=\{z\in Z:\int_0^T z(t)\,dt=0\}$ is closed in Z, and

$$\mathrm{Dim}\,\mathrm{Ker}\,L=ml=\mathrm{Codim}\,\mathrm{Im}\,L.$$

Thus, the operator L is a Fredholm mapping of index 0.

Let project operators P and Q in the following form, respectively:

$$Px=\frac{1}{T}\int_0^T x(t)\,dt,\quad x\in X,\qquad Qz=\frac{1}{T}\int_0^T z(t)\,dt,\quad z\in Z.$$

Hence,

$$\mathrm{Im}\,P=\mathrm{Ker}\,L,\qquad \mathrm{Im}\,L=\mathrm{Ker}\,Q=\mathrm{Im}(I-Q).$$

Furthermore, the generalized inverse (of L) $K_p:\mathrm{Im}\,L\to\mathrm{Ker}\,P\cap\mathrm{Dom}\,L$ reads as

$$K_p(z)=\int_0^t z(s)\,ds-\frac{1}{T}\int_0^T\int_0^t z(s)\,ds\,dt.$$

We now compute

$$QNx=\left(\frac{1}{T}\int_0^T u_1(t)\,dt,\frac{1}{T}\int_0^T u_2(t)\,dt,\dots,\frac{1}{T}\int_0^T u_l(t)\,dt\right)^T$$

and

$$K_p(I-Q)Nx=\begin{pmatrix}\int_0^t u_1^T(s)\,ds-\frac{1}{T}\int_0^T\int_0^t u_1^T(s)\,ds\,dt-\left(\frac{t}{T}-\frac{1}{2}\right)\int_0^T u_1^T(t)\,dt\\ \vdots\\ \int_0^t u_l^T(s)\,ds-\frac{1}{T}\int_0^T\int_0^t u_l^T(s)\,ds\,dt-\left(\frac{t}{T}-\frac{1}{2}\right)\int_0^T u_l^T(t)\,dt\end{pmatrix}.$$

Clearly, QN and $K_p(I-Q)N$ are continuous and $QN(\overline{\Omega})$ is bounded, where Ω is an open set in X. Then by Arzela–Ascoli theorem, we see that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact. Hence, N is L-compact on $\overline{\Omega}$.

In our proof we will use the continuation theorem of coincidence degree and graph theory. For the sake of convenience, we introduce the results concerning the coincidence degree and graph theory as follows. For more details, see [33,34].

Lemma 1 (*Mawhin's continuation theorem*). *Let L be a Fredholm mapping of index zero and N be L-compact on* $\overline{\Omega}$. *Suppose that the following conditions hold.*

(Y1) *for each* $\lambda\in(0,1)$, *every solution x of* $Lx=\lambda Nx$ *is such that* $x\notin\partial\Omega$;

(Y2) *for each* $x\in\partial\Omega\cap\mathrm{Ker}\,L$, $QNx\neq 0$ *and*

$$\deg_B\{JQN,\Omega\cap\mathrm{Ker}L,0\}\neq 0,$$

where B denotes the Brouwer degree.

Then the equation $Lx=Nx$ *has at least one solution lying in* $\mathrm{Dom}\,L\cap\overline{\Omega}$.

The following basic concepts and lemma on graph theory can be found in [33]. A digraph $\mathcal{G}=(U,E)$ contains a set $U=\{1,2,\dots,l\}$ of vertices and a set E of arcs (k,h) leading from initial vertex k to terminal vertex h. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is said to be spanning if $\mathcal{H}$ and $\mathcal{G}$ have the same vertex set. A digraph $\mathcal{G}$ is weighted if each arc (h,k) is assigned a positive weight a_{kh}. Here $a_{kh}>0$ if and only if there exists an arc from vertex h to vertex k in $\mathcal{G}$, and we call $A=(a_{kh})_{l\times l}$ as the weight matrix. The weight $W(\mathcal{G})$ of $\mathcal{G}$ is the product of the weights on all its arcs.

A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{i_1,i_2,\dots,i_s\}$ such that its set of arcs is $\{(i_k,i_{k+1}):k=1,2,\dots,s-1\}$. If $i_s=i_1$, we call $\mathcal{P}$ a directed cycle. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycles. A tree $\mathcal{T}$ is rooted at vertex k, called the root, if k is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph $\mathcal{Q}$ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle.

A digraph $\mathcal{G}$ is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. Denote the digraph with weight matrix A as $(\mathcal{G},A)$. The Laplacian matrix of $(\mathcal{G},A)$ is defined as

$$L=\begin{pmatrix}\sum_{k\neq 1}a_{1k} & -a_{12} & \cdots & -a_{1l}\\ -a_{21} & \sum_{k\neq 2}a_{2k} & \cdots & -a_{2l}\\ \vdots & \vdots & \ddots & \vdots\\ -a_{l1} & -a_{l2} & \cdots & \sum_{k\neq l}a_{lk}\end{pmatrix}.$$

A weighted digraph $(\mathcal{G},A)$ is said to be balanced if $W(\mathcal{C})=W(-\mathcal{C})$ for all directed cycles $\mathcal{C}$. Here, $-\mathcal{C}$ denotes the reverse of $\mathcal{C}$ and is constructed by reversing the direction of all arcs in $\mathcal{C}$. For a unicyclic graph $\mathcal{Q}$ with cycle $\mathcal{C}_{\mathcal{Q}}$, let $\tilde{\mathcal{Q}}$ be the unicyclic graph