Neural Networks 48 (2013) 180–194

Contents lists available at ScienceDirect

Neural Networks

journal homepage: www.elsevier.com/locate/neunet

Dynamical behaviors for discontinuous and delayed neural networks in the framework of Filippov differential inclusions[☆]

Lihong Huang[a,b,*], Zuowei Cai[a,**], Lingling Zhang[b], Lian Duan[a]

[a] College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China

[b] Hunan Women's University, Changsha, Hunan 410002, PR China

ARTICLE INFO

Article history:
Received 2 April 2013
Received in revised form 18 August 2013
Accepted 27 August 2013

Keywords:
Discontinuous neuron activation
Differential inclusions
Delayed neural networks
Krasnoselskii's fixed point theorem of set-valued maps
Positive periodic solution
Global convergence

ABSTRACT

This paper is concerned with the periodic dynamics of a class of delayed neural networks with discontinuous neural activation functions. Under the Filippov framework, the cone expansion and compression fixed point theorems of set-valued maps are successfully employed to derive the existence of the ω-periodic positive solution. However, before the discussion of the periodicity, there still remains a fundamental issue about viability to be solved due to the presence of general mixed time-delays involving both time-varying delays and distributed delays. This difficulty can be overcome by a transformation and the continuation theorem. Then, for the discontinuous and delayed neural network system with time-periodic coefficients, the uniqueness and global exponential stability of the periodic state solution are proved by using non-smooth analysis theory with generalized Lyapunov approach. Furthermore, the global convergence in measure of the periodic output is also investigated. The obtained results are a very good extension and improvement of previous works on discontinuous dynamical neuron systems with a broad range of neuron activations dropping the assumption of boundedness or monotonicity. Finally, numerical simulations are provided to illustrate the effectiveness of the theoretical results.

1. Introduction

Discontinuous dynamical neuron systems are important and frequently arise in a large number of applications (Di Marco, Forti, Grazzini, Nistri, & Pancioni, 2008; Forti, Grazzini, Nistri, & Pancioni, 2006; Forti & Nistri, 2003; Forti, Nistri, & Papini, 2005; Hopfield, 1984; Huang, Guo, & Wang, 2011; Liu & Cao, 2010; Lu & Chen, 2006; Papini & Taddei, 2005). Generally speaking, discontinuity in neural networks is caused by natural phenomenon or control actions of many interesting engineering tasks. For instance, switching in electronic circuits, systems oscillating under the effect of an earthquake, control synthesis of uncertain systems, sliding or squealing, power circuits and many others. It is worth noting that discontinuous or non-Lipschitz neuron activations have been introduced into neural network systems due to their theoretical and practical significance in recent years. Take the classic Hopfield neural networks (HNNs) as an example, the sigmoidal neuron activations would closely approach a discontinuous hard-comparator function under the standard assumption of high-gain amplifiers (see Forti & Nistri, 2003 and Hopfield, 1984). According to such a discontinuous activation, it is often advantageous to model neural networks possessing high-slope nonlinearities (Hopfield, 1984). In fact, this class of discontinuous dynamical neuron systems is usually represented by the first order differential equation with discontinuous right-hand side. In this case, the dynamics will be defined by discontinuous vector field. Moreover, there will emerge some specially interesting and important dynamical behaviors that are not captured by the continuous system, e.g., the phenomenon of convergence in finite time towards the equilibrium point or limit cycle and the presence of sliding modes along discontinuity surfaces (Cai, Huang, Guo, & Chen, 2012; Forti et al., 2005; Liu & Cao, 2010; Liu, Liu, & Xie, 2012).

As far as we know, numerous fundamental questions arise when dealing with discontinuous dynamical systems. The most basic issue we must face is the notion of solution. However, we do not know whether the classical definition of solutions is still valid for the discontinuous system. If not, what is the new framework for the solution, and how do we ensure the global existence and uniqueness? As pointed out by Cortés (2008), the existence of

☆ Research supported by 973 Program (2009CB326202), National Natural Science Foundation of China (11071060, 11371127, 11226144, 11301173) and Aid program for Science and Technology Innovative Research Team in Higher Educational Institutions of Hunan Province (XJT2008-244).

* Corresponding author at: College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China.

** Corresponding author. Tel.: +86 13467560460.

E-mail addresses: lhhuang@hnu.edu.cn (L. Huang), caizuowei01@126.com, zwcai@hnu.edu.cn (Z. Cai), linglingmath@gmail.com (L. Zhang), lianduan0906@163.com (L. Duan).

0893-6080/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.neunet.2013.08.004

a continuously differentiable solution (i.e., continuously differentiable curve whose derivative follows the direction of the vector field) is not guaranteed for the discontinuous system. Fortunately, several types of solutions defined for discontinuous system such as Filippov solutions, Carathéodory solutions, sample-and-hold solutions, Euler solutions, Krasovskii solutions and Hermes solutions have been proposed (Clarke, Ledyaev, Stern, & Wolenski, 1998; Filippov, 1988; Hermes, 1967; Krasovskii, 1963; Krasovskii & Subbotin, 1988). In particular, the concept of Filippov solution with differential inclusion framework has been accepted universally as a good one to investigate the discontinuous dynamical systems whose right-hand sides are only required to be Lebesgue measurable in the state and time variables. Such a notion of solution in the sense of Filippov is useful in the field of mathematics especially for discontinuous dynamical neuron systems. Since a Filippov solution is a limit of solutions of ordinary differential equation with a continuous right-hand side, one can model a system that is a near discontinuous system and expect that the Filippov trajectories of the discontinuous system will be close to the trajectories of the actual system (Lu & Chen, 2008). This approach is very significant in many applications, such as non-smooth analysis and variable structure control (see Aubin & Cellina, 1984, Paden & Sastry, 1987 and Utkin, 1977). From the theoretical view, under the Filippov framework, the theory of differential inclusion is a standard and effective tool to handle the problems of dynamical behaviors for differential equations with discontinuous right-hand sides (discontinuous systems). In the process of simplification of many practical problems, differential inclusion theory can relax some restrictions to the utmost extent, and does not affect the essence of the problems. In general, any mathematical model of the dynamic system containing uncertainties can be represented as a differential inclusion equation because uncertainties may lead to a suddenly change of a trajectory in such a dynamic system. Actually, the differential inclusion system is considered as a generalization of the system described by differential equation. In addition, the Filippov-framework plays a pivotal role in the analysis of non-smooth stability for discontinuous systems. The available tools involve matrix theory and a generalized Lyapunov approach based on locally Lipschitz continuous and regular (C-regular) functions. In a word, it is necessary and rewarding to study discontinuous dynamical neuron systems in the framework of Filippov differential inclusions.

It should be pointed out that, up to now, a number of researchers have investigated the dynamical behaviors of discontinuous neural networks via differential inclusions. For example, In 2003 and 2005, under the Filippov framework, Forti et al. firstly deal with the global stability of the unique equilibrium point for neural networks modeled by differential equations with discontinuous activation functions (Forti & Nistri, 2003; Forti et al., 2005). This motivated the latter studies on discontinuous neural networks (Forti et al., 2006; Huang et al., 2011; Liu & Cao, 2010; Liu, Liu, Xie et al., 2012; Lu & Chen, 2005, 2006; Qin & Xue, 2009). Nevertheless, in most of these literatures, the works are based on the assumption that the discontinuous activation functions are monotonic or bounded and neglect the effect of periodicity. As shown by Li and Huang (2009), in designing and implementing an artificial neural network, non-monotonicity might be better candidates for neuron activation functions. Song (2008) also noted that, for some applied purposes of networks (e.g., solving optimization problems in the presence of constraints such as linear, quadratic or more general programming problems), unbounded activations modeled by diode-like exponential-type functions are needed to impose constraints satisfaction. On the other hand, periodic oscillation in neural networks is an interesting and valuable dynamical behavior. Since the brain is often in periodic oscillatory or chaos state, it is of great necessity for us to investigate the periodic oscillatory phenomenon of discontinuous neural networks for understanding the function of human brain (Cai et al., 2012; Wu, 2009). Nowadays, more and more scholars observe the importance of periodic property for neural networks with discontinuous activations and have already obtained many useful results on periodic dynamical behaviors (see Huang & Guo, 2009, Huang et al., 2011, Huang, Wang, & Zhou, 2009, Liu & Cao, 2009, Lu & Chen, 2008, Wu, 2009 and their references). Here, we mention a conjecture proposed by Forti et al. (2005)that all solutions of discontinuous neural network systems converge to an asymptotically stable limit cycle (periodic solution) in the condition of periodic inputs. Subsequently, this conjecture has been extensively studied and has been extended to the situation of constant delay and variable delay (Cai et al., 2012; Liu & Cao, 2010; Wang, Huang, & Guo, 2009). However, in the case of distributed delay, there is still not much research on this conjecture.

In practice, time delays in neuron signals are inevitable owing to internal or external uncertainties. Furthermore, the existence of time delay is frequently a source of instability and oscillations for neural network systems (Marcuss & Westervelt, 1989). As discussed in Hou and Qian (1998), Huang, Ho, and Cao (2005) and Huang, Ho, and Lam (2005), in electronic implementation of artificial neural networks, the time delays are usually time variant, and sometimes vary dramatically with time because of the finite switch speed of amplifiers and faults in the electrical circuits. On the other hand, neural networks with discontinuous activations often have a spatial extent because of the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Then, there will exist either a distribution of conduction velocities along these pathways or a distribution of propagation delays over a period of time in some cases, which may cause another type of time-delays, namely, distributed time delays in neuron signals (Song & Wang, 2008; Wang, Lauria, Fang, & Liu, 2007). In these circumstances the signal propagation is not just instantaneous and cannot be modeled only with discrete delays. Therefore, for the practical design of neural networks with discontinuous neuron activations, it is of significance to consider general mixed time-delays involving both time-varying delays and distributed delays between neurons. When the time delays are introduced into the discontinuous neuron activations, we can find that the theory of functional differential inclusions (i.e., differential inclusions with memory) is used as a main tool to explore the dynamical behaviors of neural network systems modeled by time-delayed differential equations with discontinuous right-hand sides. As mentioned by Aubin and Cellina (1984), functional differential inclusions express that the velocity depends not only on the state of the system at every instant, but depends upon the history of the trajectory until this instant. We see that, in the framework of the theory of Filippov differential inclusions, delayed neural networks with discontinuous activations have been extensively studied (for example, Allegretto, Papini, & Forti, 2010, Lu & Chen, 2006, 2008, Qin & Xue, 2009 and Wang et al., 2009). Whereas, most studies mentioned above consider only a single constant time-delay and the constant delay is only an idealization of variable delay or an assumption for simplicity. Forti et al. (see Forti et al., 2005) further pointed out that it would be interesting to investigate more general discontinuous neural network models of the delay, including time-varying and distributed ones.

Inspired by the above analysis, taking the more complex and more general types of time delays into account, we will study the periodic dynamics of discontinuous neural network systems via Filippov differential inclusions. The remainder of this paper is arranged as follows. Section 2 states some preliminaries including some necessary definitions and lemmas. Section 3 describes the neural network model studied in the paper. Our main results are presented in Sections 4 and 5, where the sufficient conditions are given to guarantee the existence, uniqueness and global exponential stability for the positive periodic solution. Moreover, the