Neurocomputing 164 (2015) 293–299

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Exponential stability of a class of complex-valued neural networks with time-varying delays

Jie Pan [a,b,*], Xinzhi Liu [b], Weichau Xie [c]

[a] Department of Applied Mathematics, Sichuan Agricultural University, Chengdu 611130, PR China
[b] Department of Applied Mathematics, University of Waterloo, Ontario, Canada N2L 3G1
[c] Department of Civil & Environmental Engineering, University of Waterloo, Ontario, Canada N2L 3G1

ARTICLE INFO

Article history:
Received 3 October 2014
Received in revised form
7 February 2015
Accepted 13 February 2015
Communicated by S. Arik
Available online 20 April 2015

Keywords:
Complex-valued neural networks
Time-varying delays
Global exponential stability

ABSTRACT

This paper studies a class of complex-valued neural networks with time-varying delays. By using the conjugate system of the complex-valued neural networks and Brouwer's fixed point theorem, sufficient conditions to guarantee the existence and uniqueness of an equilibrium are obtained. Some criteria on globally exponential stability of the equilibrium of the complex-valued neural networks are also established by using a delay differential inequality. These results are easy to apply to the study of the complex-valued neural networks whether their activation functions are explicitly expressed by separating their real and imaginary parts or not. Two examples with numerical simulations are given to highlight the effectiveness of the obtained results.

1. Introduction

Recent years, there is a growing interest in studying complex-valued neural networks (CVNNs), which can be seen from a large number of relevant publications (e.g., [1–14] and the references therein). In fact, CVNNs have the potential to solve problems that cannot be solved by their real-valued counterparts. For example, in [7], the XOR problem and the detection of symmetry problem cannot be solved with a single real-valued neuron (i.e., a two-layered real-valued neural network), but they can be solved by a single complex-valued neuron (i.e., a two-layered complex-valued neural network) with the orthogonal decision boundaries, which reveals the potential computational power of complex-valued neurons. In addition, CVNNs have been found of great use in extending the scope of applications of artificial neural networks in optoelectronics, filtering, imaging, speech synthesis, computer vision, remote sensing, quantum devices, spatiotemporal analysis of physiological neural devices and systems, and artificial neural information processing [1,8–14].

In applications of neural networks, it is essential to ensure that the designed neural networks are stable. In past decades, the stability of real-valued neural networks has been widely studied and there are a large number of related publications (see, e.g., [15–24] and the references therein). It is worth mentioning that, compared with real-valued neural networks, it is more difficult to study the stability of CVNNs. Because CVNNs have more complicated properties than the real-valued counterparts. Recently, the stability of CVNNs has caught great attention (see [25–30]). For example, in [25], a class of CVNNs model on time scales was investigated and some sufficient conditions for ψ-global exponential stability were obtained. In [26], the discrete-time delayed neural networks with complex-valued linear threshold neurons were studied, and several criteria on boundedness and global exponential stability of equilibrium were obtained. In [27,28], discrete CVNNs were studied and some sufficient conditions on the existence of a unique equilibrium pattern and its global exponential stability were established. Particularly, in [29], the authors studied two types of complex-valued recurrent neural networks whose activation functions are separated into their real and imaginary parts or not. In [30], CVNNs with mixed time delays whose activation functions are expressed by separating their real and imaginary parts were studied. In [29,30], the n-dimensional complex-valued neural networks were transformed into the $2n$-dimensional real-valued ones, and some sufficient conditions ensuring the existence and uniqueness of equilibrium and its global exponential stability were achieved. However, it required the existence, continuity, and boundedness of the partial derivatives of the activation functions about the real and imaginary parts of the state variables, which limits the applications of the obtained results.

Motivated by discussion above, in this paper, we aim at dealing with the stability problem for a class of complex-valued neural networks with time-varying delays. Some sufficient conditions of

* Corresponding author at: Department of Applied Mathematics, Sichuan Agricultural University, Chengdu 611130, PR China.
E-mail address: guangjiepan@163.com (J. Pan).

http://dx.doi.org/10.1016/j.neucom.2015.02.024
0925-2312/

the existence, uniqueness, and global exponential stability of equilibrium are obtained. In these sufficient conditions, the activation functions only need to satisfy the Lipschitz condition. It removes the restrictions on the existence, continuity, and boundedness of the partial derivatives of the activation functions about their real and imaginary parts. Our method is feasible and more efficient than that in [29,30].

The rest of this paper is organized as follows. In Section 2, model description and preliminaries are given. In Section 3, several criteria are derived for the exponential stability of unique equilibrium of a class of CVNNs with time-varying delays. Then, in Section 4, two examples are given to illustrate the effectiveness of the main results. Finally, in Section 5, some conclusions are drawn.

2. Model description and preliminaries

To begin with, we would like to introduce some notations. By $\mathbb{R}$ we denote the set of real numbers. Let $z=a+ib$ be a complex number, where $i=\sqrt{-1}$, $a,b\in\mathbb{R}$, $|z|=\sqrt{a^2+b^2}$. $\bar{z}$ denotes the conjugate complex number of z, $\bar{z}=a+i(-b)$. Let $\mathbb{C}^n$ be the n $(n\geq 1)$ dimensional complex vector space.

In this paper, we consider a model of complex-valued neural networks with time-varying delays, which can be described by

$$\frac{dz_k(t)}{dt}=-d_kz_k(t)+\sum_{j=1}^{n}(w_{kj}f_j(z_j(t))+v_{kj}g_j(z_j(t-\tau_j(t))))+J_k,\quad t\geq t_0,\tag{2.1}$$

where $z_k(t)=x_k(t)+iy_k(t)$, $k,j=1,\ldots,n$. For convenience, $z_k(t)$, $x_k(t)$ and $y_k(t)$ are denoted as z_k, x_k and y_k, respectively. This model describes the continuous evolution processes of the neural networks. n is the number of units in the neural networks, $z_k(t)$ corresponds to the state variable, d_k $(d_k>0)$ represents the neuron charging time constant, $f_j(z_j)$, $g_j(z_j):\mathbb{C}\to\mathbb{C}$ are the activation functions of the neurons, w_{kj}, $v_{kj}\in\mathbb{C}$ stand for the weights of the neuron interconnections, $J_k\in\mathbb{C}$ is the external bias, and $\tau_j(t)$ $(0\leq\tau_j(t)\leq\tau)$ corresponds to the transmission delays.

Let $\mathbb{R}^n$ and $\mathbb{C}^n$ be the spaces of n-dimensional real and complex column vectors, respectively, and $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$ denote the sets of $m\times n$ real and complex matrices, respectively. For convenience, model (2.1) can be rewritten in the vector form

$$\frac{dz(t)}{dt}=-Dz(t)+Wf(z(t))+Vg(z(t-\tau(t)))+J,\quad t\geq t_0,\tag{2.2}$$

where $z(t)=(z_1(t),\ldots,z_n(t))^T$, $\frac{d}{dt}z(t)=(\frac{d}{dt}z_1(t),\ldots,\frac{d}{dt}z_n(t))^T$, $D=\mathrm{diag}(d_1,\ldots,d_n)$, $W=(w_{kj})_{n\times n}$, $V=(v_{kj})_{n\times n}$, $J=(J_1,\ldots,J_n)^T$, $f(z(t))=(f_1(z_1(t)),\ldots,f_n(z_n(t)))^T$, $g(z(t-\tau(t)))=(g_1(z_1(t-\tau_1(t))),\ldots,g_n(z_n(t-\tau_n(t))))^T$.

Definition 2.1. For any given $t_0\in\mathbb{R}$, a complex-valued function $z(t)\in C[[t_0-\tau,+\infty),\mathbb{C}^n]$ is called a solution of (2.2) through (t_0,ϕ), if $z(t)$ satisfies the initial condition

$$z(t_0+s)=\phi(s),\quad s\in[-\tau,0],\tag{2.3}$$

and Eq. (2.2) for $t\geq t_0$, denoted by $z(t,t_0,\phi)$ (or z for short). Especially, a point $Z^*\in\mathbb{C}^n$ is called an equilibrium point of (2.2), if $z(t)=z^*$ is a solution of (2.2).

Throughout this paper, we assume that, for any $\phi(s)$ $(s\in[-\tau,0])$, there exists at least one solution of model (2.2) with the initial values ϕ of model (2.3).

A conjugate system of model (2.1) is represented as

$$\frac{d\bar{z}_k(t)}{dt}=-d_k\bar{z}_k(t)+\sum_{j=1}^{n}(\overline{w}_{kj}\overline{f}_j(\bar{z}_j(t))+\overline{v}_{kj}\overline{g}_j(\bar{z}_j(t-\tau_j(t))))+\overline{J}_k,\quad t\geq t_0,\tag{2.4}$$

where $\overline{f}_j(\bar{z}_j(t))$ and $\overline{g}_j(\bar{z}_j(t-\tau_j(t)))$ are the conjugates of $f_j(z_j(t))$ and $g_j(z_j(t-\tau_j(t)))$, respectively. Similarly, we rewrite system (2.4) in the vector form

$$\frac{d\bar{z}(t)}{dt}=-D\bar{z}(t)+\overline{W}f(\bar{z}(t))+\overline{V}g(\bar{z}(t-\tau(t)))+\overline{J},\quad t\geq t_0,\tag{2.5}$$

where $\bar{z}(t)=(\bar{z}_1(t),\ldots,\bar{z}_n(t))^T$, $\overline{W}=(\overline{w}_{kj})_{n\times n}$, $\overline{V}=(\overline{v}_{kj})_{n\times n}$, $\overline{J}=(\overline{J}_1,\ldots,\overline{J}_n)^T$, $\overline{f}(\bar{z}(t))=(\overline{f}_1(\bar{z}_1(t)),\ldots,\overline{f}_n(\bar{z}_n(t)))^T$, $\overline{g}(\bar{z}(t-\tau(t)))=(\overline{g}_1(\bar{z}_1(t-\tau_1(t))),\ldots,\overline{g}_n(\bar{z}_n(t-\tau_n(t))))^T$.

Obviously, if $z(t)$ is a solution of Eq. (2.2) satisfying the initial condition $\phi(s)$ $(s\in[-\tau,0])$, then $\bar{z}(t)$ is a solution of Eq. (2.4) with initial condition $\overline{\phi}(s)$ $(s\in[-\tau,0])$. Especially, if point $z^*\in\mathbb{C}^n$ is an equilibrium point of (2.2), then $\bar{z}^*\in\mathbb{C}^n$ is an equilibrium point of (2.4).

For $A,B\in\mathbb{R}^{m\times n}$ or $\in\mathbb{C}^{m\times n}$, we define the Hadamard product or Schur product $A\otimes B=(a_{ij}b_{ij})_{m\times n}$, $|A|=(|a_{ij}|)_{n\times n}$. If $A,B\in\mathbb{R}^{m\times n}$, then $A\geq B$ $(A\leq B, A>B, A<B)$ means that each pair of corresponding elements of A and B satisfies the inequality "$\geq(\leq,>,<)$". Especially, A is called a nonnegative matrix if $A\geq 0$. For $z,\phi\in\mathbb{C}^n$, we define

$$[|z(t)|]=(|z_1(t)|,\ldots,|z_n(t)|)^T,\quad \|z(t)\|=\sqrt{\Sigma_{k=1}^n|z_k(t)|^2},$$

$$[|\phi(s)|_\tau]=(|\phi_1(s)|_\tau,\ldots,|\phi_n(s)|_\tau)^T,\quad \|\phi(s)\|_\tau=\sqrt{\Sigma_{k=1}^n|\phi_k(s)|_\tau^2},$$

where $|\phi_k(s)|_\tau=\sup_{-\tau\leq s\leq 0}\{|\phi_k(t_0+s)|\}$, $k=1,\ldots,n$.

Definition 2.2. The equilibrium point z^* of Eq. (2.2) is exponentially stable, if there exist constants $\lambda>0$ and $M\geq 1$ such that for all $t\geq t_0$ the inequality $\|z(t)-z^*\|\leq M\|\phi(s)-z^*\|e^{-\lambda(t-t_0)}$ holds.

Definition 2.3 (*[31]*). Let matrix $A=(a_{ij})_{n\times n}$ have non-positive off-diagonal elements (i.e., $a_{ij}\leq 0$, $i\neq j$); then each of the following conditions is equivalent to the statement that A is an M matrix.

(i) All the leading principal minors of A are positive.
(ii) $A=C-M$, and $\rho(C^{-1}M)<1$, where $M\geq 0$, $C=\mathrm{diag}\{c_1,\ldots,c_n\}$ and $\rho(\cdot)$ is the spectral radius of the matrix $(\cdot)$.
(iii) The diagonal elements of A are all positive and there exists a positive vector ξ such that $A\xi>0$ or $A^T\xi>0$.

Assumption 1. $f_j(\cdot)$, $g_j(\cdot)$ $(j=1,\ldots,n)$ satisfy the Lipschitz continuity condition in the complex domain, that is, for $j=1,\ldots,n$, there exist positive constants l_j^f and l_j^g, such that, for any $z^1,z^2\in\mathbb{C}$, we have

$$|f_j(z^1)-f_j(z^2)|\leq l_j^f|z^1-z^2|,\quad |g_j(z^1)-g_j(z^2)|\leq l_j^g|z^1-z^2|,$$

where l_j^f and l_j^g are called Lipschitz constants.

Remark 2.1. In this paper, it only requires that complex-valued activation functions be globally Lipschitz, as many researchers have required in the study of real-valued neural networks (see [23,24,32–37]).

Remark 2.2. It should be pointed out that, in [29,30], it were required that the activation functions can be expressed by separating real and imaginary parts as

$$f_j(z_j)=f_j^R(x_j,y_j)+if_j^I(x_j,y_j),\quad j=1,\ldots,n,$$

where $z_j=x+iy_j$, $x_j,y_j\in\mathbb{R}$, $f_j^R(x_j,y_j)$ (or f_j^R) and $f_j^I(x_j,y_j)$ (or f_j^I) denote real and imaginary parts of $f_j(z_j)$, $j=1,\ldots,n$, respectively.

The existence, continuity, and boundness of $\frac{\partial f_j^R}{\partial x_j}$, $\frac{\partial f_j^R}{\partial y_j}$, $\frac{\partial f_j^I}{\partial x_j}$, and $\frac{\partial f_j^I}{\partial y_j}$ $(j=1,\ldots,n)$ are required to ensure the stability of the system considered (see Theorem 2 in [29] and Theorem 1 in [30]). These conditions are quite strict, but not necessary. In fact, if $f_j^R(x_j,y_j)$,