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Almost periodic solutions of impulsive neural networks at non-prescribed moments of time

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ABSTRACT

In this paper, we address a new model of neural networks related to the discontinuity phenomena which is called impulsive recurrent neural networks with variable moments of time. Sufficient conditions for existence and uniqueness of exponentially stable almost periodic solution are investigated. An example is given to illustrate our theoretical results.

1. Introduction

Impulsive neural networks (see, for example [2,4,11,12,15,16,20,21,24,25,27–31]) have been enormously developed issuing from the fact that in implementation of electronic networks, the state of the networks is subject to instantaneous perturbations and experiences abrupt change at certain moments, which may be caused by switching phenomenon, frequency change or other sudden noises. On the other hand, studies on neural dynamical systems not only involve stability and periodicity, but also involve other dynamic behaviors such as almost periodicity, chaos and bifurcation. If one considers long-term dynamical behaviors, the periodic parameters often turn out to experience certain perturbations, that is, parameters become periodic up to a small error. Thus, almost periodic oscillatory behavior is considered to be more accordant with reality. Although it is of great importance in real life applications, the generalization to almost periodicity has been rarely studied in the literature; see [13,14,17,19,21,24–26,29,32,33]. In the present paper we introduce a new class of neural networks related to the discontinuity phenomena which appear at non-prescribed moments of time. The main purpose of introducing this class is that the moments of discontinuity θ_k are arbitrary in $\mathbb{R}$. There are many papers dealing with impulsive neural networks with *fixed moments* of time [2,11,12,15,16,20,21,24,25,27–31] and the references therein. It deserves to be mentioned that there have been no results on impulsive neural networks with variable moments of time as well as almost periodic solutions for these networks.

The main novelty of the paper is to investigate sufficient conditions ensuring the existence and uniqueness of almost periodic solution. To solve the problem, we should develop the technique of the reduction of the considered system to system with fixed moments of impulses. That is, *B*-equivalence method, which was studied for bounded domain in the phase space [1,3,5–10]. Equations with nonfixed moments of discontinuity create a great number of opportunities for theoretical inquiry as well as theoretical challenges. The proposed new involvements have an important role for the real world problems. Exceptional practical interest is connected with discontinuities, which appear at non-prescribed moments of time.

2. *B*-equivalence systems

Let $\mathbb{Z}$ and $\mathbb{R}$ be the sets of integers and real numbers. Consider the following impulsive recurrent neural networks with variable moments of time:

$$\begin{aligned} x_i'(t) &= -a_i(t)x_i(t)+\sum_{j=1}^{m} b_{ij}(t)f_j(x_j(t))+c_i(t),\\ \Delta x_i|_{t=\theta_k+\tau_k(x)} &= d_{ik}x_i+I_{ik}(x), \end{aligned} \tag{2.1}$$

where $a_i(t)>0,\ i=1,2,\ldots,m,\ k\in\mathbb{Z},\ x\in\mathbb{R}^m, t\in\mathbb{R},\ \{d_{ik}\}$ is a bounded sequence such that $(1+d_{ik})\neq 0,\ i=1,2,\ldots,m,\ k\in\mathbb{Z},$

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$\tau_k(x)$ are positive real valued continuous functions defined on $\mathbb{R}^m$, $k \in \mathbb{Z}$. Moreover, the sequence θ_k satisfies the following condition $\theta_k < \theta_{k+1}$, $|\theta_k| \to +\infty$ as $|k| \to \infty$.

In system (2.1), $x_i(t)$ denotes the membrane potential of the unit i at time t; the continuous functions $f_j(\cdot)$ represent the measures of activation to its incoming potentials of the unit j at time t; b_{ij} corresponds to the synaptic connection weight of the unit j on the unit i; c_i signifies the external bias or input from outside the network to the unit i; a_i is the rate with which the ith unit will reset its potential to the resting state in isolation when it is disconnected from the network and external inputs. It will be assumed that $a_i, b_{ij}, c_i, I_{ik} : \mathbb{R}^m \to \mathbb{R}^m$ are continuous functions, where $i, j = 1, 2, \ldots, m$, $k \in \mathbb{Z}$.

The following assumptions will be needed throughout the paper:

(A1) there exists a Lipschitz constant $\ell > 0$ such that

$$|\tau_k(x) - \tau_k(y)| + |f_i(x) - f_i(y)| + |I_{ik}(x) - I_{ik}(y)| \le \ell |x - y|$$

and $|\tau_k(x)| < \ell$ for all $x, y \in \mathbb{R}^m$, $i = 1, 2, \ldots, m$, $k \in \mathbb{Z}$;

(A2) there exists a positive number $\theta \in \mathbb{R}$ such that $\theta_{k+1} - \theta_k \ge \theta$ holds for all $k \in \mathbb{Z}$ and the surfaces of discontinuity $\Gamma_k : t = \theta_k + \tau_k(x)$, $k \in \mathbb{Z}$, satisfy the following conditions:

$$\theta_k + \tau_k(x) < \theta_{k+1} + \tau_{k+1}(x), \quad |\theta_k| \to +\infty \text{ as } |k| \to \infty,$$

$$\tau_k((E + D_k)x + I_k(x)) \le \tau_k(x), \quad x \in \mathbb{R}^m,$$

where E is an $m \times m$ identity matrix and

$$D_k = diag(d_{1k}, \ldots, d_{mk}) = \begin{pmatrix} d_{1k} & 0 & \ldots & 0 \\ 0 & d_{2k} & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & d_{mk} \end{pmatrix} \quad \text{and}$$

$$I_k = \begin{pmatrix} I_{1k} \\ I_{2k} \\ \ldots \\ I_{mk} \end{pmatrix};$$

(A3) $\ell(k_1 h + k_2) < 1$.

For the sake of convenience, we adopt the following notations in the sequel:

$$k_1 = \max_{1 \le i \le m} \sup_{t \in \mathbb{R}} \left(|a_i(t)| + \ell \sum_{j=1}^{m} |b_{ji}(t)| \right) < +\infty,$$

$$k_2 = \max_{1 \le i \le m} \sup_{t \in \mathbb{R}} \left(\sum_{j=1}^{m} |b_{ji}(t)| |f_i(0)| + |c_i(t)| \right) < +\infty,$$

$$k_3 = \max_{1 \le i \le m} \sup_{t \in \mathbb{R}} \left(\sum_{j=1}^{m} |b_{ji}(t)| \right) < +\infty, \quad k_4 = \max_{k \ge 1}(|J_{ik}(0)|) < +\infty.$$

From local existence theorem [3, Theorem 5.2.1], a solution of (2.1) exists. By virtue of Theorem 5.3.1 in [3] and assumptions (A2)–(A3), every solution $x(t)$, $\|x(t)\| \le h$ of (2.1) intersects each surface of discontinuity $\Gamma_k : t = \theta_k + \tau_k(x)$, $k \in \mathbb{Z}$, at most once. Furthermore, by the proof of Theorem 5.2.4 in [3], continuation of solutions of ordinary differential equation

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j(x_j(t)) + c_i(t) \tag{2.2}$$

and the condition $|\theta_k| \to +\infty$ as $|k| \to \infty$, one can find that every solution $x(t) = x(t, t_0, x^0)$, $(t_0, x^0) \in \mathbb{R} \times \mathbb{R}^m$, of (2.1) is continuable on $\mathbb{R}$. That is to say, the interval of existence is a whole real line.

For a fix $k \in \mathbb{Z}$, let $x^0(t) = x(t, \theta_k, x^0)$ be a solution of the system of ordinary differential equations (2.2). Denote by $t = \xi_k$ the time when the solution of (2.2) intersects the surface of discontinuity $\Gamma_k : t = \theta_k + \tau_k(x(\xi_k))$, $k \in \mathbb{Z}$. Suppose that $x^1(t) = x(t, \xi_k, (E + D_k) x^0(\theta_k) + I_{ik}(x^0(\xi_k)))$ is also a solution of (2.2). Next, we define a mapping $J_{ik}(x) : \mathbb{R}^m \to \mathbb{R}^m$ such that $J_{ik}(x) = x^1(\theta_k) - (E + D_k)x$ (Fig. 1 illustrates the procedure of the contraction of the map J_{ik}) and construct a system of impulsive differential equations with fixed moments, which has the form

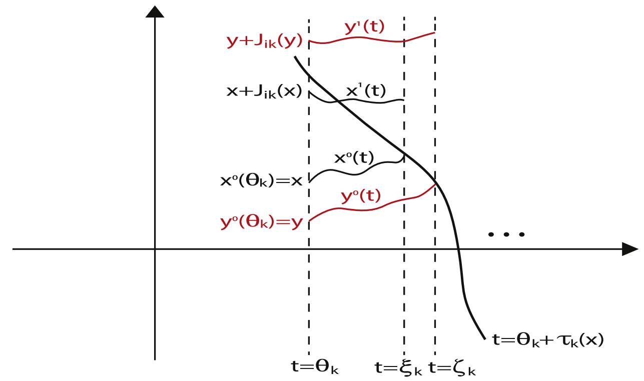

Fig. 1. The procedure of the construction of the map J_{ik}.

$$y_i'(t) = -a_i(t)y_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j(y_j(t)) + c_i(t),$$
$$\Delta y_i|_{t=\theta_k} = d_{ik} y_i + J_{ik}(y), \tag{2.3}$$

where $a_i(t) > 0$, $i = 1, 2, \ldots, m$, $k \in \mathbb{Z}$.

A difficulty in investigation of a system (2.1) is that the discontinuities of distinct solutions are not, in general, the same. To investigate the asymptotic properties of solutions of Eq. (2.1), we introduce the following concepts. In what follows, we give the techniques of B-topology and B-equivalence which were introduced and developed in [3] for the systems of differential equations with variable moments of time. For detailed discussion, we refer to reader to the book [3].

We denote $\mathcal{PC}(J; \mathbb{R}^m)$, $J \subset \mathbb{R}$, the space of all piecewise continuous functions $\varphi : J \to \mathbb{R}^m$ with points of discontinuity of the first kind θ_k, $k \in \mathbb{Z}$ and which are continuous from the left.

Let $x(t)$ be a solution of Eq. (2.1) on $\mathcal{U}$ ($\mathcal{U}$ can be an interval, a real half-line, or the real line $\mathbb{R}$).

Definition 2.1. A solution $y(t)$ of (2.3) is said to be in the ε-neighborhood of a solution $x(t)$ if:

(i) the measure of the symmetrical difference between the domains of existence of these solutions does not exceed ϵ;
(ii) discontinuity points of $y(t)$ are in ε-neighborhoods of discontinuity points of $x(t)$;
(iii) for all $t \in \mathcal{U}$ outside of ε-neighborhoods of discontinuity points of $x(t)$ the inequality $\|x(t) - y(t)\| < \epsilon$ holds.

The topology defined by ε-neighborhoods of piecewise continuous solutions will be called the B-topology. It is easily seen that it is Hausdorff topology. Topologies and metrics for spaces of discontinuous functions were introduced and developed in [5,9,18].

For any $u, v \in \mathbb{R}$ we define the oriented interval $\widehat{[u, v]}$ as

$$\widehat{[u, v]} = \left\{ \begin{array}{ll} [u, v] & \text{if } u \le v \\ [v, u] & \text{otherwise} \end{array} \right\}. \tag{2.4}$$

Definition 2.2. Systems (2.1) and (2.3) are said to be B-equivalent, if for any solution $x(t)$ of (2.1) defined on an interval $\mathcal{U}$ with the discontinuity points ξ_k, $k \in \mathbb{Z}$, there exists a solution $y(t)$ of system