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Asymptotic boundedness for stochastic coupled systems on networks with Markovian switching

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ABSTRACT

In this paper, a novel class of stochastic coupled systems on networks with Markovian switching is presented. In such model, the white noise, the color noise and the coupling between different vertices of the network are taken into account. Focusing on the boundedness problem, this paper employs the Lyapunov method, some graph theory and the method of *M*-matrix to establish some simple and easy-verified boundedness criteria. These criteria can directly show the link between the graph structure of the network and the dynamics of coupled systems. Finally, stochastic coupled van der Pol's equations with Markovian switching are used to demonstrate our findings. Meanwhile, two numerical examples are also provided to clearly show the influence of coupled structure on the boundedness of coupled systems.

1. Introduction

Recently, coupled systems on networks (CSNs) become increasingly significant, since it can model the evolution of complex networks of diverse nature, including coupled lasers in physics, neuronal networks in biology and communication networks in technology [1–7]. Therein, many researchers devoted themselves to the study of dynamical properties for CSNs such as stability, boundedness and synchronization (see [8–13] and the references therein). It is well-known that boundedness of solutions plays a critical role in investigating the existence of the periodic solution, the uniqueness of equilibrium, kinds of global stability problems, chaos control and synchronization, and so on [14–16]. Therefore, the boundedness analysis of CSNs is a significant problem and thus has been extensively investigated [17–20]. For instance, in [17], Liu and Chen discussed the boundedness problem of the y-coupled Lorenz systems with or without controllers. In [19], Fu et al. considered the uniform boundedness of the global solutions for a strongly coupled three-species cooperating model. Moreover, the relation between the boundedness criterion and the topology property for some CSNs was also concerned in [20].

In practice, the dynamic structure of CSNs is prone to various types of stochastic disturbances including white noise and color noise. This fact demonstrates that it is critical in many fields to discover whether the presence of such noise will significantly affect the systems [21–24]. In recent years, more attention was paid to the CSNs under the perturbation of white noise. For instance, in [25], a systematic method for constructing the global Lyapunov function for stochastic coupled systems on networks (SCSNs) was proposed by combining some graph theory and the Lyapunov method. However, to the authors' best knowledge, there exist few results reported to the CSNs with the perturbation of color noise.

In this paper, our intention is to analyze the asymptotic boundedness for stochastic coupled systems on a network with Markovian switching (SCSNMS) as follows:

$$\begin{aligned} \mathrm{d}x_k(t) = {}& \left[f_k(x_k(t), t, r(t)) + \sum_{h \in \mathbb{L}} H_{kh}(x_k(t), x_h(t), t, r(t)) \right] \mathrm{d}t \\ & + \left[g_k(x_k(t), t, r(t)) + \sum_{h \in \mathbb{L}} N_{kh}(x_k(t), x_h(t), t, r(t)) \right] \mathrm{d}W(t), \quad k \in \mathbb{L}. \end{aligned} \tag{1}$$

In such model, both white noise and color noise are taken into account. Here, white noise refers to the generalized mean-square derivative of Brownian motion $W(\cdot)$. Color noise is modeled by a finite-state Markovian switching $r(\cdot)$ since it can always be described as a random switching between two or more environmental regimes. Obviously, SCSNMS (1) could be described on a digraph and the detailed description will be shown in the next section.

In general, the boundedness analysis for (1) is a formidable task. On the one hand, both the coupling structure and the presence of Markovian switching and white noise will affect the boundedness of

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(1). It should be noted that the main challenge in studies of dynamical network (1) is to relate the structure of the graph of the network to its dynamics [26]. On the other hand, the available approach for showing the asymptotic boundedness of some systems with Markovian switching is essentially based on the common Lyapunov method or variations of the same framework. However, it is quite difficult to construct an appropriate Lyapunov function and this is a well known disadvantage of Lyapunov method. Also, the boundedness criterion of many SCSNMSs obtained from this method is often too conservative for practical applications.

The above discussion clearly shows that analyzing the boundedness problem of SCSNMS (1) is not trivial and some new method must be introduced. Recently, a new graph-theoretic approach to the method of global Lyapunov functions was proposed by [27,28]. The authors used graph theory to explore the global stability for a general deterministic coupled system on networks, and the sufficient conditions for the global asymptotical stability were given. Furthermore, in [29,30], the global stability for several classes of multi-group models in mathematical epidemiology was effectively investigated by the method. Moreover, the technique was developed for SCSNs with or without Markovian switching, and many stability criteria were obtained in [25,31,32]. Meanwhile, this new graph-theoretic method has also been applied on the boundedness of some concrete stochastic coupled networks: stochastic coupled van der Pol oscillators with time-varying delayed coupling [33] and stochastic Cohen–Grossberg neural networks with Markovian switching [15]. However, till now, there is no boundedness result for the general SCSNMS (1) based on this new approach.

Motivated by the above discussions, the aim of this paper is to develop a general approach to construct the Lyapunov function for SCSNMS (1) based on the Lyapunov method and graph theory in [25,27], and furthermore, to give the sufficient conditions which ensure the asymptotic boundedness for SCSNMS (1). Compared to the existing results (e.g., [25,27] and other results), contributions of this paper are as follows. Firstly, this paper focuses on the boundedness, while existing results are mainly about the stability. In fact, except for the stability property, boundedness is also one of the foundational concepts of dynamical systems. What is more, the property of boundedness of solutions plays an important role in many investigations. Secondly, in the model of our paper, three important factors including the white noise, the color noise and the coupling between different vertices of the network are taken into account. But the existing results always only consider one or two factors, which may not be so consistent with the fact. Finally, in addition to adopting new graph-theoretic method in existing results, the method of M-matrix is also used to get the criterion about boundedness.

This paper is arranged as follows. In Section 2, some useful notations, lemma and problem formulation are stated. Section 3 is devoted to giving a criterion on the asymptotic boundedness in pth moment in terms of the Lyapunov method and graph theory. Moreover, to obtain some more simple and easy-verified asymptotic boundedness conditions, another criterion is investigated by combining the method of M-matrix. In Section 4, to demonstrate the application of the proposed generalized approaches, the asymptotic boundedness for some coupled van der Pol's equations with Markovian switching on networks is studied and the numerical simulations are also given, which is followed by conclusions in Section 5.

2. Preliminaries

In this section, we need to state some useful notations, lemma and definition associated with asymptotic boundedness in pth moment. Moreover, the problem formulation is also presented.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \ge 0}$ satisfying the usual conditions, and $W(\cdot)$ be a one-dimensional Brownian motion defined on the space. The mathematical expectation with respect to the given probability measure $\mathbb{P}$ is denoted by $\mathbb{E}(\cdot)$. Let $r(t)$ be a right-continuous Markov chain taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$ and $\gamma_{ij} \ge 0$ is the transition rate from i to j while $\gamma_{ii} = -\sum_{i \ne j} \gamma_{ij}$. Denote $|\cdot|$ as the Euclidean norm for vectors or the trace norm for matrices. Through this paper, we shall use the notations $\mathbb{Z}^+ = \{1, 2, \ldots\}$, $\mathbb{L} = \{1, 2, \ldots, l\}$, $m = \sum_{k \in \mathbb{L}} m_k$, where $m_k \in \mathbb{Z}^+$, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0,\ i = 1, 2 \ldots, n\}$ and $\mathbb{R}^n_0 = \mathbb{R}^n - \{0\}$.

To describe SCSNMS (1) on a digraph, it is necessary to show the basic concepts and notations on graph theory [34]. A digraph $\mathcal{G}$ is weighted if each arc (h,k) is assigned a positive weight a_{kh}. Define the weighted matrix $A = (a_{kh})_{l \times l}$ whose entry a_{kh} equals the weight of arc (h,k) if it exists, and 0 otherwise. Denote the directed graph with weight matrix A as $(\mathcal{G}, A)$. The Laplacian matrix of $(\mathcal{G}, A)$ is defined as $L = (p_{kh})_{l \times l}$, where $p_{kh} = -a_{kh}$ for $k \ne h$ and $p_{kh} = \sum_{j \ne k} a_{kj}$ for $k = h$. Here we show a lemma which will be used in the proof of our main theorem.

Lemma 1. *Assume that $l \ge 2$ and $c_k(i)$ denotes the cofactor of the kth diagonal element of Laplacian matrix of $(\mathcal{G}, A(i))$, $A(i) = (a_{kh}(i))_{l \times l}$. Then the following identity holds:*

$$\sum_{k,h \in \mathbb{L}} c_k(i) a_{kh}(i) F_{kh}(x_k, x_h, t, i) = \sum_{\mathcal{Q} \in \mathbb{Q}} W(\mathcal{Q}) \sum_{(s,r) \in E(C_{\mathcal{Q}})} F_{rs}(x_r, x_s, t, i).$$

Here $i \in \mathbb{S}$, $F_{kh}(x_k, x_h, t, i)$ $(k, h \in \mathbb{L})$ are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A(i))$, $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $C_{\mathcal{Q}}$ denotes the directed cycle of $\mathcal{Q}$. In particular, if $(\mathcal{G}, A(i))$ is strongly connected, then $c_k(i) > 0$ for $k \in \mathbb{L}$.

Proof of this lemma is similar to which appeared in Theorem 2.2 in [27]. It just needs some obvious modifications, so we omit it here for brevity.

Now we use digraph $\mathcal{G} = (M, E)$ to describe SCSNMS (1), in which $\mathcal{G}$ contains a set $M = \{1, 2, \ldots, l\}$ of vertices and a set E of arcs (k,h) leading from initial vertex k to terminal vertex h. Firstly, in the kth vertex, assign an m_k-dimensional stochastic differential equation with Markovian switching (refer to the kth vertex system):

$$dx_k(t) = f_k(x_k(t), t, r(t))\, dt + g_k(x_k(t), t, r(t))\, dW(t), \quad t \ge 0,$$

where $x_k(t) \in \mathbb{R}^{m_k}$, $f_k, g_k : \mathbb{R}^{m_k} \times \mathbb{R}^1_+ \times \mathbb{S} \to \mathbb{R}^{m_k}$. Secondly, let H_{kh}, $N_{kh} : \mathbb{R}^{m_k} \times \mathbb{R}^{m_h} \times \mathbb{R}^1_+ \times \mathbb{S} \to \mathbb{R}^{m_k}$ represent the influence of vertex h on vertex k and $H_{kh} = N_{kh} = 0$ if there exists no arc from h to k in $\mathcal{G}$. Then by coupling the l vertex systems by H_{kh} and N_{kh}, i.e. replacing f_k and g_k by $f_k + \sum_{h \in \mathbb{L}} H_{kh}$ and $g_k + \sum_{h \in \mathbb{L}} N_{kh}$, respectively, SCSNMS (1) is thus obtained.

As usual, throughout this paper, we suppose that both f_k and g_k satisfy certain conditions so that (1) has a unique global solution, denoted by $x(t, x_0, r_0) = (x_1(t), \ldots, x_l(t))^T$, where superscript T means transform. In the paper we mainly consider the asymptotic boundedness of (1), whose definition is given as follows.

Definition 1 (*Mao and Yuan [22]*). SCSNMS (1) is said to be asymptotically bounded in pth moment $(p > 0)$, if there is a positive constant M such that

$$\limsup_{t \to \infty} \mathbb{E}|x(t, x_0, r_0)|^p \le M$$

for all $(x_0, r_0) \in \mathbb{R}^m \times \mathbb{S}$. Particularly, it is said to be asymptotically bounded in mean square when $p = 2$.

Under some simple conditions, it can be shown that $x(t, x_0, r_0)$ will never reach zero whenever $x_0 \ne 0$ (see [22], Lemma 5.1).