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A projection neural network with mixed delays for solving linear variational inequality ☆

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ABSTRACT

This paper presents a projection neural network with discrete delays and distributed delays (i.e. mixed delays) for solving linear variational inequality (LVI). By the Lyapunov theory and the linear matrix inequality (LMI) approach, the neural network is proved to be globally exponentially convergent to the solution of LVI. Compared with existing neural networks for solving LVI, the proposed one features the ability of solving a class of non-monotone LVI. One numerical example is provided to illustrate the effectiveness and the satisfactory performance of the neural network.

1. Introduction

Linear variational inequality (LVI): find $x^* \in \Omega$ such that

$$(Mx^* + q)^T (x - x^*) \geq 0, \quad \forall x \in \Omega, \tag{1}$$

where $M \in R^{n \times n}$, $q \in R^n$, $x = (x_1, \ldots, x_n)^T \in R^n$ and the feasible domain $\Omega = \{x \in R^n \,|\, l_i \leq x_i \leq h_i, i = 1, \ldots, n\}$. LVI and nonlinear variational inequality (in which $Mx + p$ is replaced by a nonlinear function $F(x) : R^n \to R$) are of crucial importance in mathematical programming, which have numerous applications in science and engineering [1]. During past decades, due to its inherent nature for parallel computation and real-time applications, neural networks for solving LVI and related optimization problems have been widely investigated [2–9]. For example, Xia et al. developed the projection neural network to solve LVI and nonlinear variational problems [2,3]. Hu et al. employed the projection neural network for "solving" pseudomonotone variational inequalities [5]. It is noted that, most of the neural networks mentioned above are under two stringent assumptions: (a) the LVI must be monotone or strictly monotone, or equivalently, M is positive semidefinite or positive definite to guarantee their convergence to the optimal solution; (b) the neurons communicate and respond without time delays.

Time delays inevitably occur during the signal communication among the neurons, which may lead complex dynamical behavior of network by oscillation or instability [14–18]. In recent decade, there has been an increasing interest in the study of employing delayed neural networks to solve LVI (1) and related optimization problems [10–13]. For example, in consideration of the transmission delays occurring in different parts of neural networks, Liu et al. [10] and Yang et al. [11] proposed two kinds of DNN for solving linear projection equations and quadratic programming problems, respectively. Based on the neural network model in [11], Cheng et al. presented another delayed neural network in [12] for solving LVI problems and further extended the results to the case of time-varying delays [13]. It should be noted that the delays in all above-mentioned delayed neural networks are restricted to the case of discrete delays. However, a neural network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety axon sizes and lengths. It is desirable to model them by introducing distributed delays. In other words, it is often the case that the neural network model possesses both discrete and distributed delays, i.e. mixed delays [19]. In these years, dynamics analysis of neural networks with mixed delays have attracted considerable research interests [20–24]. However, to the best of the authors' knowledge, most results have been concentrated in the stability analysis and there is no existing neural network with mixed delays being employed to solve LVI or related optimization problems.

In this paper, we present a projection neural network with mixed delays for solving LVI (1). By the Lyapunov theory, the proposed neural network is proved to be globally exponentially convergent to the solution of LVI (1). By the proposed LMI approach, the positive semidefiniteness or positive definiteness condition on M has been relaxed. As a consequence, the proposed

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neural network can also solve a class of nonmonotone LVI. Finally, the satisfactory performance of the proposed neural network is demonstrated by a numerical example.

2. Problem formulation and preliminaries

Notions: The superscript T denotes the transpose of a matrix or vector. I denotes an identity matrix with compatible dimensions. For a vector δ, $\|\delta\| = \sqrt{\sum_{i=1}^{n} \delta_i^2}$ denotes the l_2-norm of δ. For a matrix V, $\lambda_M(V)$ and $\lambda_m(V)$ denote the maximum and the minimum eigenvalue of V, respectively. $\|V\| = \sqrt{\lambda_M(V^T V)}$ represents the matrix spectral norm of V. For a square matrix V, $V \geq 0$ and $V > 0$ means that it is positive semidefinite and positive definite matrix, respectively.

Based on the projection theorem [1], it follows that $x^* \in \Omega$ is a solution of the LVI defined in (1) if and only if it satisfies the following projection equation:

$$x = P_\Omega[x - \alpha(Mx + q)], \tag{2}$$

where $\alpha > 0$ is a constant and P_Ω: $R^n \to \Omega$ is a projection operator defined by $P_\Omega(x) = [P_\Omega(x_1), \ldots, P_\Omega(x_n)]^T$, and

$$P_\Omega(x_i) = \begin{cases} l_i, & x_i < l_i, \\ x_i, & l_i \leq x_i \leq h_i, \\ h_i, & x_i > h_i. \end{cases} \tag{3}$$

In view of the equivalent formulation of the LVI in (2), the following projection neural network was developed for solving LVI (1) in [2–4]:

$$\frac{dx(t)}{dt} = \lambda\{P_\Omega[x(t) - \alpha(Mx(t) + q)] - x(t)\}, \tag{4}$$

where $\lambda > 0$ is the scaling factor and P_Ω is defined in (3).

Note that x^* is a solution of LVI (1) if and only if it is the equilibrium point of the above neural network. In [2–4], the characteristics of the neural network (4) have been fully analyzed. It was shown that, if M is symmetric and $M \geq 0$, the neural network (2) is stable and globally convergent to the solution of LVI (1), and if $M > 0$ the neural network (2) is globally exponentially convergent to the solution of LVI (1). That is to say, the neural network (2) can solve the monotone LVI problem.

Taking the time delay effect into consideration, we propose a projection neural network with mixed delays for solving LVI (1) as follows:

$$\frac{dx(t)}{dt} = -(2 + \tau_2)x(t) + P_\Omega[x(t) - \alpha(Mx(t) + q)] + x(t - \tau_1) + \int_{t-\tau_2}^{t} x(s)\, ds, \quad t \in (t_0, +\infty),$$

$$x(t) = \phi(t), \quad t \in [t_0 - \tau_M, t_0], \tag{5}$$

where $\tau_1 \geq 0$, $\tau_2 \geq 0$, $\tau_M = \max(\tau_1, \tau_2)$, $\phi(t) \in C([t_0 - \tau_M, t_0], R^n)$ and $C([t_0 - \tau_M, t_0], R^n)$ denotes the set of all continuous vector-valued functions from $[t_0 - \tau_M, t_0]$. It is easy to see that x^* is an equilibrium point of (5) if and only if x^* is the optimal solution of problem (1). Consequently, the neural network (5) can be employed to solve LVI (1).

In order to obtain the main results, a definition and two lemmas are introduced first as follows.

Definition 1 (*Lien et al. [20]*). The equilibrium point x^* of the delayed projection neural network defined by (5) is said to be globally exponentially stable with convergence rate $\xi > 0$ if there exist positive constants ξ and γ such that

$$\|x(t) - x^*\| \leq \gamma \cdot e^{-\xi(t - t_0)}, \quad \forall t \geq t_0.$$

Lemma 1 (*Liu et al. [10]*). *The projection operator P_Ω satisfies the following inequality for any $x, y \in R^n$:*

$$\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\|$$

and

$$(P_\Omega(x) - P_\Omega(y))^T (P_\Omega(x) - P_\Omega(y)) \leq (P_\Omega(x) - P_\Omega(y))^T (x - y).$$

Lemma 2 (*Gu [25]*). *For any symmetric positive definite matrix $M_0 \in R^{n\times n}$, scalar $\rho > 0$ and vector function $\omega : [0, \rho] \to R^n$, such that the integrations concerned are well defined, the following inequality holds*:

$$\rho\left(\int_0^\rho \omega^T(s) M_0 \omega(s)\, ds\right) \geq \left(\int_0^\rho \omega(s)\, ds\right)^T M_0 \left(\int_0^\rho \omega(s)\, ds\right).$$

3. Main results

To prove the exponential convergence of the neural network (5), we reformulate the neural network (5) as follows:

$$\frac{dx(t)}{dt} = -(2 + \tau_2)x(t) + P_\Omega(Ax(t) + b) + x(t - \tau_1) + \int_{t-\tau_2}^{t} x(s)\, ds, \tag{6}$$

where $A = (a_{ij})_{n\times n} = I - \alpha M$ and $b = (b_1, b_2, \ldots, b_n)^T = -\alpha q$.

Then, let x^* be the equilibrium point of the neural network (5) and $a_i = [a_{i1}, a_{i2}, \ldots, a_{in}]$. By coordinate transformation $u(t) = x(t) - x^*$, we get the following system from (6):

$$\frac{du(t)}{dt} = -(2 + \tau_2)u(t) + f(Au(t)) + u(t - \tau_1) + \int_{t-\tau_2}^{t} u(s)\, ds, \tag{7}$$

where $f(Au(t)) = [f_1(a_1 u(t)), f_2(a_2 u(t)), \ldots, f_n(a_n x(t))]^T$ and $f_i(a_i u(t)) = P_\Omega^i[a_i(u(t) + x^*) + b_i] - P_\Omega^i(a_i x^* + b_i), i = 1, 2, \ldots, n$.

Clearly, x^* is globally exponentially stable for system (5) if and only if the zero solution of system (7) is globally exponentially stable.

Theorem 1. *If there exist positive-definite symmetric matrices $P \in R^{n\times n}$, $Q \in R^{n\times n}$, $H \in R^{n\times n}$ and $Z \in R^{n\times n}$, positive-definite diagonal matrices $D = diag\{d_1, d_2, \ldots, d_n\}$ and $L = diag\{l_1, l_2, \ldots, l_n\}$, real matrices T_1, T_2, T_3, T_4 and a constant $k > 0$ such that the following LMI (8) holds, then the equilibrium point of the neural network (5) is globally exponentially stable.*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & -T_1 & P \\ * & \Xi_{22} & \Xi_{23} & T_3^T & -T_2 & T_4 \\ * & * & \Xi_{33} & \Xi_{34} & 0 & DA \\ * & * & * & \Xi_{44} & 0 & T_3 \\ * & * & * & * & \Xi_{55} & 0 \\ * & * & * & * & * & \Xi_{66} \end{bmatrix} < 0, \tag{8}$$

where $*$ are entries readily inferred by symmetry and

$$\Xi_{11} = (2k - 4 - 2\tau_2)P + Q + \tau_2 H + T_1 + T_1^T,$$
$$\Xi_{12} = P - T_1 - T_2^T,$$
$$\Xi_{13} = P - (2k + 2 + \tau_2)A^T D + A^T L - (2 + \tau_2)T_4^T,$$
$$\Xi_{14} = -(2 + \tau_2)T_3^T,$$
$$\Xi_{22} = -e^{-2k\tau_1} Q - T_2 - T_2^T,$$
$$\Xi_{23} = A^T D + T_4^T,$$
$$\Xi_{33} = DA + A^T D - 2L + T_4 + T_4^T,$$
$$\Xi_{34} = T_3^T - T_4,$$
$$\Xi_{44} = \tau_1 Z - T_3 - T_3^T,$$