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# Asymmetric neural network synchronization and dynamics based on an adaptive learning rule of synapses

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ABSTRACT

An adaptive learning rule of synapses was proposed for a general asymmetric neural network. Its feasibility was proved by the Lasalle principle. Numerical simulation results show that synaptic connection weight can converge to an appropriate strength and the network comes to synchronization. Furthermore, ISI (inter-spike interval) of synchronization orbit in neural network has a typical period doubling bifurcation. It is a further improvement compared with bifurcation of the traditional single neuron model, which promotes our understanding of neuron population activities.


## 1. Introduction

With the diverse applications of synchronization in engineering field, network synchronization studies have been rapidly advanced [1–7]. By controlling some system state variables, the system can reach the synchronization. Moreover, in the biophysical experiment, scientists found synchronous discharges of neurons also existed in the brain [8–10]. This discharge pattern may be related to some functions of the brain. Synchronization in neural network plays a significant role in the understanding of the brain.

Synchronization studies of nervous system began with two coupling neurons, and then went to network. The topological structures of these networks considered are mainly regular networks, such as chain, grid and ring [4–7]. Furthermore, the small-world network for simulating the nervous system is also involved [11–13]. However, they are enormously different from the real nervous system due to their regularity. Because of the impact of dynamic system in engineering application, in many studies another problem like symmetrical coupling exists. In other words, the synaptic connection is mutual, while it does not necessarily appear in the real nervous system. The improvement of these two problems has been made in this study.

Now that synchronous discharge may be connected with some functions of the brain, a natural question is how to achieve the synchronization in nervous system. By external control of some system state variables, synchronization can be attained from some studies on engineering dynamic system [14–16], while the external control is not required for the synchronization of nervous system. Due to the plasticity of synapse between neurons, synchronization can be achieved in the nervous system by synaptic adaptive learning. We propose an adaptive learning rule of synapses to simulate plasticity dependent on spike timing, demonstrating the feasibility of the algorithm by theoretical proof and simulation.

Another question is what the synchronous orbit is if it exists. Most results are concerned with dynamic system orbit on single neuron or two coupling neurons [1–3]. Period doubling bifurcation was also simulated on single neuron or two coupling neurons, but it should occur to neural network in fact. We find period doubling bifurcation to chaos also appears in network, going further than the former results.

## 2. Adaptive synaptic learning algorithm and theoretical analysis

Considering a normal network consisting of n neurons, its topology connection matrix $A=(a_{ij})\in R^{n\times n}$. Among $a_{ij}=1$ when there is a synapse from $j$th neuron to the $i$th neuron; for no connection and $a_{ij}=0, i=j$. Due to the difference between neural network and a general dynamic system, synaptic connection is asymmetrical and thus $a_{ij}$ is not necessarily equal to $a_{ji}$. $C(t)=(c_{ij}(t))\in R^{n\times n}$ means the strength of synaptic connection.

Synchronization of neural network means membrane potential synchronization. We assume the first state variable of dynamic system is membrane potential. Then the single neuron model is

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$\dot{X}^i=(f_1(X^i),f_2(X^i),f_3(X^i),...,f_m(X^i))^T$, where $X^i=(x_1^i,x_2^i,\cdots,x_m^i)^T$ indicates the state variable, and $x_1^i$ is membrane potential. Furthermore, the model of network is

$$\dot{X}^i=(f_1(X^i)+\sum_{j=1}^{n}a_{ij}c_{ij}(t)(x_1^j-x_1^i),f_2(X^i),f_3(X^i),...,f_m(X^i))^T$$

$X=(X^1,X^2,...,X^n)\in R^{m\times n}$ is the state of this neural system.

The following adaptive learning rule of synapses was designed for neuron $i$ and neuron $j$.

(1) Calculate synaptic learning probability $P_{ij}(t)=\exp(-1/(|x_1^j-x_1^i|T(t)+\varepsilon))$, where $T(t)=T_0/(1+\alpha t)$ and $\varepsilon$ is an arbitrarily small positive number to make sure that no zero appears in denominator during the computational process.
(2) Generate a random number $r$. It is subject to uniform distribution in [0,1].
(3) Compare learning probability and random number. If $P_{ij}>r$, it is believed that synapses learning occur and the strength is modified; otherwise, there is no change. The learning rule is $\dot{c}_{ij}=\eta(x_1^j-x_1^i)^2$, where $\eta$ represents learning rate.

The algorithm sets a probability judgment, that is, learning happens only when $P_{ij}>r$. When presynaptic and postsynaptic neurons are subthreshold activities, $|x_1^j-x_1^i|$ is very small. Therefore, the synaptic learning probability $P_{ij}$ is also very small, or even close to zero. The unlikely occurrence of $P>r$ makes the probability of learning very small. When asynchronous discharge appears successively, $|x_1^j-x_1^i|$ becomes larger and then $P_{ij}$ becomes larger. The larger $P_{ij}$ becomes, the more possibly synaptic learning appears. These two properties accord with the learning rule of synapses in the real biophysical experiment. Since learning is dependent on spike times, the sequential occurrence of action potentials significantly strengthens connection [14]. Under Hebb principles, strength change of synapses does not easily occur in subthreshold activities.

$T(t)$ is a decreasing function and $\lim_{t\to+\infty}T(t)=0$, which is similar to cooling function in the simulated annealing algorithm. So other forms can also be chosen, and synaptic learning more easily happens in relatively high temperature.

$\lim_{t\to+\infty}P_{ij}(t)=\lim_{t\to+\infty}\exp(-1/(|x_1^j-x_1^i|T(t)+\varepsilon))=e^{-1/\varepsilon}$, which indicates when time $t$ is large enough, the learning probability $P_{ij}$ becomes smaller than any positive number given if $\varepsilon$ is chosen small enough. Then the occurrence of learning is made into a small probability event. This design well agrees with the biophysical results because network synaptic learning has the limitation and network will turn to be mature.

When time step tends toward zero, $\dot{c}_{ij}(t)=\eta P_{ij}(t)(x_1^j-x_1^i)^2$.

Then Lasalle principle is used to prove feasibility of synchronization of asymmetric irregular neural network by this adaptive algorithm as follows:

**Theorem 1.** (*Lasalle*) [17] *For dynamic system $\dot{X}=f(X),f:R^n\to R^n$ is a $C^1$ continuous map. If function $V(X):R^n\to R^+$ exists, for each $X\in R^n$, $\dot{V}(X)\le 0$ and definition $E:=\{X\in R^n:\dot{V}(X)=0\}$. Given that B is the largest invariant set in E, then each bounded solution will converge to B when $t\to+\infty$.*

**Hypothesis 1.** $f_1(X^i)$ satisfies Lipschitz condition about first state variable (membrane potential), $\|f_1(X^i)-f_1(X^j)\|\le L\|x_1^i-x_1^j\|$.

For a general neural network with $n$ neurons, we study the error dynamic system.

$$\dot{e}_{ij}=\dot{x}_1^i-\dot{x}_1^j,i,j=1,2,...,n$$

$$\dot{c}_{ij}(t)=\eta P_{ij}(t)(x_1^j-x_1^i)^2$$

**Theorem 2.** $B=\{(e_{ij},c_{ij})\in R^{2n^2}:e_{ij}=0,c_{ij}=\hat{c}_{ij},i,j=1,2,...,n\}$. *For error dynamic system, the orbit will converge to B with any initial states, that is $e_{ij}=x_1^i-x_1^j\to 0$, $c_{ij}\to\hat{c}_{ij},t\to+\infty$.*

**Proof.** For error system, we define Lyapunov function

$$V(t)=\sum_{i,j=1}^{n}V_{ij}(t)=\sum_{i,j=1}^{n}\left\{1/(2e_{ij}^2)+1/(2\eta)\left(\sum_{l=1}^{n}c_{ij}-M/\delta\right)^2 +1/(2\eta)\left(\sum_{l=1}^{n}c_{ji}-M/\delta\right)^2\right\}\ge 0$$

where $\delta=e^{-1/\varepsilon}$, $c_0=\max_{i,j}\{\lim_{t\to+\infty}c_{ij}(t)\}$. Constant M is subject to $M\ge 2c_0n^2+L$. Then,

$$\begin{aligned}
e_{ij}\sum_{j=1}^{n}a_{ij}c_{ij}e_{ji}&\le|e_{ij}|\sum_{j=1}^{n}|c_{ij}e_{ji}|=|e_{ij}|(|c_{i1}e_{1i}|+|c_{i2}e_{2i}|+\cdots|c_{in}e_{ni}|)\\
&\le c_{i1}(e_{ij}^2+e_{1i}^2)/2+c_{i2}(e_{ij}^2+e_{2i}^2)/2+\cdots c_{in}(e_{ij}^2+e_{ni}^2)/2\\
&=\frac{1}{2}e_{ij}^2\sum_{j=1}^{n}c_{ij}+\frac{1}{2}\sum_{j=1}^{n}(c_{ij}e_{ji}^2)\\
&\le\frac{1}{2}e_{ij}^2\sum_{j=1}^{n}c_{ij}+\frac{1}{2}\sum_{j=1}^{n}c_{ij}\sum_{j=1}^{n}e_{ij}^2
\end{aligned}$$

So,

$$\begin{aligned}
\frac{dV_{ij}(t)}{dt}&=\frac{d}{dt}\left\{\frac{1}{2}e_{ij}^2+\frac{1}{2\eta}\left(\sum_{j=1}^{n}c_{ij}-M/\delta\right)^2+\frac{1}{2\eta}\left(\sum_{i=1}^{n}c_{ji}-M/\delta\right)^2\right\}\\
&=e_{ij}\dot{e}_{ij}+\frac{1}{\eta}\left(\sum_{j=1}^{n}c_{ij}-\frac{M}{\delta}\right)\sum_{j=1}^{n}\dot{c}_{ij}+\frac{1}{\eta}\left(\sum_{i=1}^{n}c_{ji}-\frac{M}{\delta}\right)\sum_{i=1}^{n}\dot{c}_{ji}\\
&=e_{ij}(f_1(X^i)-f_1(X^j))+e_{ij}\sum_{j=1}^{n}a_{ij}c_{ij}e_{ji}+e_{ji}\sum_{i=1}^{n}a_{ji}c_{ji}e_{ij}\\
&+\frac{1}{\eta}\sum_{j=1}^{n}c_{ij}\sum_{j=1}^{n}\eta P_{ij}(t)e_{ij}^2-\frac{1}{\eta}\frac{M}{\delta}\sum_{j=1}^{n}\eta P_{ij}(t)e_{ij}^2\\
&+\frac{1}{\eta}\sum_{i=1}^{n}c_{ji}\sum_{i=1}^{n}\eta P_{ji}(t)e_{ij}^2-\frac{1}{\eta}\frac{M}{\delta}\sum_{i=1}^{n}\eta P_{ji}(t)e_{ij}^2\\
&\le Le_{ij}^2+\frac{1}{2}e_{ij}^2\sum_{j=1}^{n}c_{ij}+\frac{1}{2}\sum_{j=1}^{n}c_{ij}\sum_{j=1}^{n}e_{ij}^2+\frac{1}{2}e_{ij}^2\sum_{i=1}^{n}c_{ji}+\frac{1}{2}\sum_{i=1}^{n}c_{ji}\sum_{i=1}^{n}e_{ij}^2\\
&+\sum_{j=1}^{n}c_{ij}\sum_{j=1}^{n}e_{ij}^2-M\sum_{j=1}^{n}e_{ij}^2+\sum_{i=1}^{n}c_{ij}\sum_{i=1}^{n}e_{ij}^2-M\sum_{i=1}^{n}e_{ij}^2\qquad(\delta\le P_{ij}\le 1)\\
&\le L\sum_{j=1}^{n}e_{ij}^2+2\sum_{j=1}^{n}c_{ij}\sum_{j=1}^{n}e_{ij}^2+2\sum_{i=1}^{n}c_{ji}\sum_{i=1}^{n}e_{ij}^2-M\sum_{j=1}^{n}e_{ij}^2-M\sum_{i=1}^{n}e_{ij}^2\\
&=(L+2\sum_{j=1}^{n}c_{ij}-M)\sum_{j=1}^{n}e_{ij}^2+(\sum_{i=1}^{n}c_{ji}-M)\sum_{i=1}^{n}e_{ij}^2\\
&\le 0
\end{aligned}$$

We can get $dV(t)/dt=\sum_{i,j=1}^{n}(dV_{ij}(t)/dt)\le 0$.

$\dot{V}(t)=0$ if and only if $e_{ij}=0$, $i,j=1,2...n$. That is to say, $B$ is the largest invariant set in $E$. By theorem 1, we can get the conclusion that the orbit will converge to B with any initial states, that is $e_{ij}=x_1^i-x_1^j\to 0,c_{ij}\to\hat{c}_{ij},t\to+\infty$.

By the proof procedure, we find that this adaptive synchronization approach can be applied to large-scale irregular network, especially asymmetric network. Furthermore, it can be applied to biological neural networks because its properties well agree with some experiment results. The synaptic connection weights will converge to appropriate values finally by the adaptive learning. In fact, the network can also achieve synchronization under weights