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The general critical analysis for continuous-time UPPAM recurrent neural networks ☆

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ABSTRACT

The uniformly pseudo-projection-anti-monotone (UPPAM) neural network model, which can be considered as the unified continuous-time neural networks (CNNs), includes almost all of the known CNNs individuals. Recently, studies on the critical dynamic behaviors of CNNs have drawn special attentions due to its importance in both theory and applications. In this paper, we will present the analysis of the UPPAM network under the general critical conditions. It is shown that the UPPAM network possesses the global convergence and asymptotical stability under the general critical conditions if the network satisfies one quasi-symmetric requirement on the connective matrices, which is easy to be verified and applied. The general critical dynamics have rarely been studied before, and this work is an attempt to gain a meaningful assurance of general critical convergence and stability of CNNs. Since UPPAM network is the unified model for CNNs, the results obtained here can generalize and extend the existing critical conclusions for CNNs individuals, let alone those non-critical cases. Moreover, the easily verified conditions for general critical convergence and stability can further promote the applications of CNNs.

1. Introduction

The two basic elements of a recurrent neural network (RNN) are the synaptic connections among the neurons and the nonlinear activation functions deduced from the input–output properties of the involved neurons. For applications such as associative memory, synaptic connections among the neurons are designed to encode the memories we hope to recover. The activation functions are assumed to capture the complex, nonlinear response of neurons of the brain. For different purpose of simulations and applications, both of them are preassigned before use. So understanding their properties is very important, and especially exploring the characteristics of the activation functions is quite crucial to determine the performance of the RNNs. For the commonly used RNN individuals, the activation functions are monotonically nondecreasing and saturated. To study and apply RNNs only based on such two features are far from enough. To overcome the non-thorough descriptions of activation functions, many special cases of activation functions have been brought forward, resulting in many different RNNs individuals. Furthermore, in order to obtain more useful results of RNNs, e.g., the convergence and stability of those individuals, additional strict requirements are unavoidable to impose on the networks for the lack of in-depth descriptions on the activation functions. Obviously, since those individuals are studied separately, it is inevitable that there exist large numbers of redundancy of analysis for those individual models. In order to reduce the superabundance, establishing a harmonization methodology is a challenging work.

In [16], Xu and Qiao put forward two novel concepts: uniformly anti-monotone and the pseudo-projection properties of the activation functions, which discover more essential characteristics other than the nondecreasing and bounded properties of the commonly used activation functions. It is shown that the proposed uniformly pseudo-projection anti-monotone (UPPAM) operator can embody most of the activation operators (the precise definition of uniformly pseudo-projection-anti-monotone operator will be given in Section 2), e.g., nearest-point projection, linear saturating operator, signum operator, symmetric multi-valued step operator, multi-threshold operator, and winner-take-all operator. Thus, the UPPAM operator can be considered as a framework of formalizing most of the activation operators of RNNs.

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In this paper, we use the concept of UPPAM operators to establish a unified model for continuous-time RNNs. Let us consider the following continuous-time UPPAM RNNs model:

$$\tau\frac{dx(t)}{dt} = -x(t)+AG(Wx(t)+q)+b, \quad x_0 \in \mathbb{R}^N \tag{1}$$

where $x(t)=(x_1(t),x_2(t),\ldots,x_N(t))^T$ is the neural network state, $G=(g_1,g_2,\ldots,g_N)^T$ is the nonlinear activation operator deduced from all the activation functions g_i, and G owns the uniformly pseudo-projection-anti-monotone property. Both A and W are the connective weight matrices, b, q are two fixed external bias vectors and τ is the state feedback coefficient. The form of model (1) includes two basic kinds of continuous-time RNNs [17], i.e., the static RNNs and the local field RNNs. Furthermore, as proved in [16], most activation operators are special cases of the UPPAM operator. So, model (1) can be considered as a unified model of continuous-time RNNs and can include almost all of the existing continuous-time RNNs specials [4], e.g., Hopfield-type neural networks, Brain-State-in-a-Box neural networks, Recurrent Back-propagation neural networks, Mean-field neural networks, Bound-constraints Optimization Solvers, Convex Optimization Solvers, Recurrent Correlation Associative Memories neural networks, and Cellular neural networks. In addition, since model (1) owns the essential characteristics of the activation functions, i.e., the uniformly anti-monotone as well as the pseudo-projection properties, it can be expected that the analysis of model (1), especially the dynamics analysis can give more in-depth results and provide the unified and concise characterization of the continuous-time RNNs models. The main purpose of this paper will focus on discovering some essential global convergence and stability for the unified model (1), i.e., the critical convergence and stability.

For RNNs, one difficult problem of dynamics analysis lies in the critical analysis. Define a discriminant matrix

$$S(\Gamma,P)=\Gamma P-\frac{\Gamma AW+(\Gamma AW)^T}{2},$$

where Γ is a positive definite diagonal matrix, P is a diagonal matrix defined by the network, and W and A are the weight matrices. If there exist a positive definite diagonal matrix Γ, such that $S(\Gamma,2\Lambda-B)>0$ (i.e., $S(\Gamma,2\Lambda-B)$ is positive definite), where Λ and B are the anti-monotone and pseudo-projection constant matrices of the network (the definitions of them are given in Section 2), then RNNs have exponential stability [4]. Many stability results have been achieved for RNNs individuals under various specifications of $S(\Gamma,2\Lambda-B)>0$ (typically, when $S(\Gamma,2\Lambda-B)>0$ is an M-matrix), and they are called as the non-critical dynamical analysis [1]. On the other hand, if there exists a positive definite diagonal matrix Γ such that $S(\Gamma,V)$ is negative definite, here $V=diag\{r_1,r_2,\ldots,r_N\}$ with each $r_i>0$ being the maximum inversely Lipschitz constant of g_i (i.e., for all $s,t\in\mathcal{R}^N$, $|g_i(t)-g_i(s)| \geq r_i|t-s|$), then RNNs are globally exponentially unstable [7,1]. Since $S(\Gamma,2\Lambda-B)>0$ is the sufficient condition on the globally exponential stability of RNNs, and $S(\Gamma,V)\geq 0$ is the necessarycondition for RNNs to be globally stable, it is quite natural to explore the gap between $S(\Gamma,2\Lambda-B)\leq 0$ (i.e., $S(\Gamma,2\Lambda-B)$ is negative semi-definite) and $S(\Gamma,V)\geq 0$ (i.e., $S(\Gamma,V)$ is positive definite). Such a gap is called the general critical condition, and the dynamics analysis of RNNs under such condition is referred to as the general critical dynamics analysis.

For any application and practical design of RNNs, such as pattern recognition, associative memories, or as optimization solvers, the convergence and stability of RNNs are both prerequisite. For instance, when an RNN is used in associative memory or pattern recognition, any pattern we hope to store has to be an equilibrium point of the RNN. In addition, to ensure that each stored pattern can be retrieved even with noises, each equilibrium point must possess the stability. When the RNN is employed as an optimization solver, the possible optimal solutions correspond to the equilibrium of the RNN, and the convergence of the RNN is a guarantee of finding the optimal solutions. Since the general critical conditions can be considered essentially as the distinct region of stability and non-stability of RNNs, studying the general critical dynamic behaviors of an RNN can find broad applications.

Recently, due to the difficulty in the dynamical analysis of RNNs for general critical conditions, most of the studies on critical analysis have been focused on the special critical conditions, i.e., considering the asymptotic behaviors of RNNs under the condition that $S(\Gamma,2\Lambda-B)\geq 0$ (this is because $S(\Gamma,2\Lambda-B)>0$ is already known to be globally exponential stable and $S(\Gamma,2\Lambda-B)=0$ is a special case of the general critical condition). Even for this special critical condition, there only exist a few results since the analysis is much more difficult than the dynamics analysis under the non-critical condition that $S(\Lambda,L)>0$. In [15], the globally exponential stability of a static neural network with projection operator (a special kind of UPPAM operator) has been proven under the condition that I–W is nonnegative (which is a special case of $S(\Gamma,2\Lambda-B)\geq 0$). The special critical convergence of a static neural network model with nearest point projection activation operator (special case of projection operator) on a region defined by the network has been achieved in [1] when W is quasi-symmetric. Some general critical stability conclusions for the static and the local field continuous-time RNNs with projection activation operators have been achieved in [2], but they require the network to satisfy one bounded matrix norm. In [4], for the presented unified continuous-time RNNs, namely, UPPAM RNNs, the special critical global convergence is obtained with some bound requirements on the defined nonlinear norm, but such requirements cannot be verified easily in applications. In [5], some improvements on dynamics analysis of the UPPAM networks have been obtained, while they are still under the special critical conditions.

In the present paper, we give some solutions on how to assure the convergence and stability under the general critical conditions. By applying the energy function method and LaSalle invariance principle to the unified continuous-time RNNs model (1), we obtain the global convergence and asymptotic stability under some general critical conditions, that is, $S(\Gamma,2\Lambda-B)+\Psi$ is positive definite for one diagonal matrix Ψ. The results only require the network to satisfy some quasi-symmetric conditions on the connection matrices. Since the conclusions obtained here are for the unified RNNs model under the general critical conditions, they can sharpen and generalize, to a large extent, the latest critical results given by [1,2,4,5,15], and they can further be extended to those non-critical conclusions (see, e.g., [6–14,18–25] and the references quoted there). Furthermore, they can be applied directly to many individual RNN models mentioned above. They can be widely applied to solve the linear variational inequality and many other optimization problems, etc. Therefore, the study here provides an insight on the unified continuous-time RNNs model with critical analysis.

2. Preliminaries

For the activation operator G, the domain, range and fixed-point set of G are respectively defined by $\mathbf{D}(G)$, $\mathbf{R}(G)$ and $\mathbf{F}(G)$, and $\mathbf{D}(G)=\mathbf{R}(G)\subseteq\mathbb{R}^N$. Assume that $\mathbb{R}^N$ is embedded with Euclidean norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$.

For any $x=(x_1,x_2,\ldots,x_N)^T\in\mathbf{D}(G)$, write

$$G(x)=(g_1(x),g_2(x),\cdots,g_N(x))^T, \quad \forall x\in\mathbf{D}(G)$$

G is said to be diagonal if $g_i(x)=g_i(x_i)$ holds for each $i=1,2,\ldots,N$.