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Asymptotic stability for neural networks with mixed time-delays: The discrete-time case[☆,☆☆]

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ABSTRACT

This paper is concerned with the stability analysis problem for a new class of discrete-time recurrent neural networks with mixed time-delays. The mixed time-delays that consist of both the discrete and distributed time-delays are addressed, for the first time, when analyzing the asymptotic stability for discrete-time neural networks. The activation functions are not required to be differentiable or strictly monotonic. The existence of the equilibrium point is first proved under mild conditions. By constructing a new Lyapnuov–Krasovskii functional, a linear matrix inequality (LMI) approach is developed to establish sufficient conditions for the discrete-time neural networks to be globally asymptotically stable. As an extension, we further consider the stability analysis problem for the same class of neural networks but with state-dependent stochastic disturbances. All the conditions obtained are expressed in terms of LMIs whose feasibility can be easily checked by using the numerically efficient Matlab LMI Toolbox. A simulation example is presented to show the usefulness of the derived LMI-based stability condition.

1. Introduction

In the past few decades, recurrent neural networks (RNNs) have received intensive interest due to their wide applications in a variety of areas including such as pattern recognition, associative memory and combinational optimization. Dynamical behaviors (e.g. stability, instability, periodic oscillatory and chaos) of the neural networks are known to be crucial in applications. For instance, if a neural network is employed to solve some optimization problems, it is highly desirable for the neural network to have a unique globally stable equilibrium. Therefore, stability analysis of neural networks has received much attention and various stability conditions have been obtained.

Time delay is an inherent feature of signal transmission between neurons, and becomes one of the main sources for causing instability and poor performances of neural networks (see e.g. Arik (2000), Gao, Lam, and Wang (2006) and Gao, Lam, and Chen (2006)). According to the way it occurs, time-delay can be classified as two types: discrete and distributed. Discrete time-delay is relatively easier to be identified in practice and, therefore, stability analysis for RNNs with discrete delays has been an attractive subject of research in the past few years. Various sufficient conditions, either delay-dependent or delay-independent, have been proposed to guarantee the global asymptotic or exponential stability for the RNNs, see e.g. Cao and Song (2006), Song and Cao (2006), Wang, Liu, and Liu (2005) and Wang, Liu, Li, and Liu (2006) for some recent publications. On the other hand, due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths, a neural network usually has a spatial nature. Therefore, it is necessary to introduce continuously *distributed delays* over a certain duration of time such that the distant past has less influence compared with the recent behavior of the state (Principle, Kuo, & Celebi, 1994; Tank & Hopfield, 1987). Recently, the global stability analysis problem for general RNNs with *both discrete and distributed delays* (or called *mixed time-delays*) has received increasing research attention and many relevant results have been reported in the literature, see e.g. Liu, Wang, and Liu (2006), Wang et al. (2005, 2006) and the references therein.

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It should be pointed out that, to date, almost all results concerning dynamics analysis problems for RNNs with mixed time-delays have been on continuous-time models. In implementing and applications of neural networks, however, *discrete-time* neural networks play a more important role than their continuous-time counterparts in today's digital world. If one wants to simulate or compute the continuous-time neural network, it is essential to formulate the discrete-time analogue so as to investigate the dynamical characteristics (Mohamad & Naim, 2002; Mohamad & Gopalsamy, 2003; Stuart & Humphries, 1996). In the past few years, various stability criteria have been proposed for discrete-time neural networks (DNNs) in the literature, see e.g. Hu and Wang (2006), Wang and Xu (2006), Xiong and Cao (2005), Yuan, Hu, and Huang (2005), Zhao and Wang (2006) and Zou and Zhou (2006) for DNNs without time delays and Chen, Lu, and Liang (2006), Liang, Cao, and Lam (2005), Liang, Cao, and Ho (2005) and Xiang, Yan, and Wang (2005) for DNNs with discrete time-delays. Note that pioneering work has been carried out in Mohamad (2008) for preserving exponential stability in discrete-time analogues of artificial neural networks with distributed delays.

It has now been well recognized that, in implementations of neural networks, stochastic disturbances are nearly inevitable owing to thermal noise in electronic devices. It has also been shown that certain stochastic inputs could destabilize a neural network. Therefore, the stability analysis problem for discrete-time stochastic neural networks with time-delays becomes more significant from the practical point of view, and initial results related to this problem has recently been published in Liu, Wang, and Liu (2007) and the references therein. Unfortunately, so far, the stability analysis problem for *discrete-time stochastic* neural networks with *mixed time-delays* has not been fully investigated yet and remains challenging. The major difficulty stems from the question that how to represent the distributed time-delays in the discrete-time domain and then establish a unified framework to handle both the discrete and distributed time-delays. The main purpose of the present research is to make the first attempt to shorten such a gap.

In this paper, we study the asymptotic stability problem for a new class of *discrete-time* stochastic neural networks with both discrete and distributed time-delays. We first deal with the deterministic neural network. The existence of the equilibrium point is proved under mild conditions on the activation functions, where neither differentiability nor monotonicity is needed. By constructing a new Lyapnuov–Krasovskii functional, a linear matrix inequality (LMI) approach is developed to establish sufficient conditions for the discrete-time neural networks to be globally asymptotically stable. As an extension, we then consider the stability analysis problem for the same class of neural networks but with state-dependent stochastic disturbances. All the conditions obtained are expressed in terms of LMIs whose feasibility can be easily checked by using the numerically efficient Matlab LMI Toolbox. Note that LMIs can be easily solved by using the Matlab LMI toolbox, and no tuning of parameters is required (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994). A simulation example is presented to show the usefulness of the derived LMI-based stability condition.

Notations: The notations are quite standard. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes matrix transposition and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). In symmetric block matrices, we use an asterisk "$*$" to represent a term that is induced by symmetry. For vector or matrix z, $z \succeq 0$ means that each entry of z is nonnegative. I_n is the $n \times n$ identity matrix. $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) means the largest (respectively, smallest) eigenvalue of A. Matrices, if not explicitly specified, are assumed to have compatible dimensions. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

2. Problem formulation

Consider the following n-neuron discrete-time neural network with discrete and distributed delays of the form:

$$u_i(k+1) = a_i u(k) + \sum_{j=1}^{n} b_{ij}\hat{f}_j(u_j(k)) + \sum_{j=1}^{n} c_{ij}\hat{g}_j(u_j(k-\tau(k))) + \sum_{j=1}^{n} d_{ij}\sum_{m=1}^{+\infty}\mu_m \hat{h}_j(u_j(k-m)) + J_i, \quad i = 1, 2, \ldots, n, \tag{1}$$

or, in an equivalent vector form

$$u(k+1) = Au(k) + B\hat{F}(u(k)) + C\hat{G}(u(k-\tau(k))) + D\sum_{m=1}^{+\infty}\mu_m \hat{H}(u(k-m)) + J \tag{2}$$

where $u(k) = (u_1(k), u_2(k), \ldots, u_n(k))^{\mathrm{T}}$ is the neural state vector, $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\}$ is the state feedback coefficient matrix; the $n \times n$ matrices $B = [b_{ij}]_{n\times n}$, $C = [c_{ij}]_{n\times n}$ and $D = [d_{ij}]_{n\times n}$ are, respectively, the connection weight matrix, the discretely delayed connection weight matrix and distributively delayed connection weight matrix. The positive integer $\tau(k)$ denotes the time-varying delay satisfying

$$\tau_m \leq \tau(k) \leq \tau_M, \quad k \in \mathbb{N}, \tag{3}$$

where τ_m and τ_M are known positive integers. In (2), $\hat{F}(u(k)) = [\hat{f}_1(u_1(k)), \hat{f}_2(u_2(k)), \ldots, \hat{f}_n(u_n(k))]^{\mathrm{T}}$, $\hat{G}(u(k)) = [\hat{g}_1(u_1(k)), \hat{g}_2(u_2(k)), \ldots, \hat{g}_n(u_n(k))]^{\mathrm{T}}$ and $\hat{H}(u(k)) = [\hat{h}_1(u_1(k)), \hat{h}_2(u_2(k)), \ldots, \hat{h}_n(u_n(k))]^{\mathrm{T}}$ denote the neuron activation functions. The constant vector $J = [J_1, J_2, \ldots, J_n]^{\mathrm{T}}$ is the exogenous input and $\mu_m (m = 1, 2, \ldots)$ are scalar constants.

Remark 1. The model (1) or (2) is quite general and can be seen as the discrete analog of the following well-studied continuous-time RNN with mixed time delay:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + BF(u(t)) + CG(u(t-\tau(t))) + D\int_{-\infty}^{t} k(t-s)H(u(s))\mathrm{d}s + J.$$

The activation functions are usually assumed to be continuous, differentiable, monotonically increasing and bounded, such as the sigmoid-type of function. However, in many electronic circuits, the input–output functions of amplifiers may be neither monotonically increasing nor continuously differentiable, hence nonmonotonic functions can be more appropriate to describe the neuron activation in designing and implementing an artificial neural network. In this paper, we make following assumptions for the neuron activation functions.

Assumption 1. For $i \in \{1, 2, \ldots, n\}$, the neuron activation functions $\hat{f}_i(\cdot)$, $\hat{g}_i(\cdot)$ and $\hat{h}_i(\cdot)$ in (1) or (2) are continuous and bounded.