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Matrix measure based stability criteria for high-order neural networks with proportional delay

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A B S T R A C T

In this paper, the stability is discussed for high-order neural networks with proportional delay. The proportional delay is a time-varying unbounded delay and different from the constant delay, bounded time-varying delays and distributed delays. Based on Lyapunov method, matrix measure and generalized Halanay inequality, a criterion is obtained to ensure the *p*th exponential stability of high-order neural networks with proportional delay. The result can be extended to the neural networks with proportional delay or multiple proportional delays. The obtained results are simple, effective and easy to be verified. The simulating examples are exploited to illustrate the improvement and advantages of the obtained results in comparison with some existing results.

1. Introduction

In recent years, delayed neural networks have attracted considerable attention and many important results on them have been given (see [1–22], and references therein). In fact, time delays are ubiquitous for a variety of axon sizes and lengths and the finite signal propagation time, which may cause instability, oscillation and bifurcation, etc. Hence, time delays are introduced into the models of neural networks. In most researches, time delays are usually assumed to be bounded. However, time delays of the practical systems may be unbounded. For example, in the models of human brains, neural networks have memory function, and time delays provide information of history and the entire history affects the present, so in this case delays are inevitably unbounded. Another example, in Web quality of service (QoS) routing decision, proportional delay (i.e. delay is proportional to the time) is usually required [23]. Therefore, it is significative to research the unbounded delays neural networks, such as [14,17–22]. In [17], authors address the stability of neural networks with unbounded time-varying delays and with bounded Lipschitz continuous activation functions. Chen and Wang in [18] point out that along the increasing of the delays, the stability of the equilibrium point will change from exponential-stable to power-stable, log-stable, and log–log-stable.

In this paper, we consider the stability of neural networks with proportional delay. Proportional delay is proportional to the time and a special unbounded time-varying delay. The proportional delay systems are important mathematical models, which often rise in some fields such as physics, biology systems and control theory. Proportional delay equation research has attracted many scholars' interests [24–26]. Since proportional delay equations are different from other delayed equations, thus most results of the stability for delayed neural networks cannot be directly applied to neural networks with proportional delay, see [11,20–22]. Zhou in [21] studied the dissipativity of cellular neural networks with proportional delays. In [22], Zhou gave some criteria to ensure global uniform asymptotic stability of cellular neural networks with multi-proportional delays.

However, the general neural networks are lower-order and shown to have limitations such as limited capacity, see [27–29], which led many researchers to use neural networks with high order connections. The high-order neural networks have greater storage capacity, stronger approximation property, faster convergence rate, and higher fault tolerance than lower-order neural networks. Recently, various results on stability of high-order delayed neural networks are obtained, see [30–34]. Cao et al. in [31] studied a class of high-order bidirectional associative memory neural networks with constant time delays. In [34], authors discussed the global stability for high-order Hopfield-type neural networks with bounded time-varying delays. To the best of our

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knowledge, few authors have considered the stability for high-order neural networks with proportional delay.

Motivated by the above discussions, it is meaningful to study the stability for high-order neural networks with proportional delay, which is our aim in this paper. By using Lyapunov method, matrix measure and generalized Halanay inequality, a sufficient condition is acquired to ensure the pth exponential stability of high-order neural networks with proportional delay. This result can be extended to the neural networks with proportional delay or multiple proportional delays. The obtained results are simple, effective and easy to be verified. Some simulating examples illustrate the improvement and advantages of the results in comparison with some existing results. The main contributions of this paper lie in three aspects: (i) our model of neural networks is high-order and contains proportional delay, which is equivalent to one of the high-order neural networks with time-varying coefficients and constant delay; (ii) our results are about pth ($p \in \{1, 2, \infty\}$) exponentially stable with uniform form, which are different with others only about $p=1$, $p=2$ or $p=\infty$; (iii) using matrix measure avoids constructing the complex Lyapunov function, the obtained results are more simple and easy to be verified.

The rest of this paper is organized as follows. In Section 2, the discussed model is proposed and some preliminaries are briefly outlined. In Section 3, some criteria are derived for stability of the proposed neural networks by matrix measure and generalized Halanay inequality. In Section 4, some numerical examples are provided to show the effectiveness of the obtained results. Some conclusions are finally drawn in Section 5.

2. Model formulation and some preliminaries

Consider the high-order neural network with proportional delay as follows:

$$\dot{u}_i(t) = -d_i u_i(t) + \sum_{j=1}^{n} [a_{ij} f_j(u_j(t)) + b_{ij} g_j(u_j(qt))] + \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk} g_j(u_j(qt)) g_k(u_k(qt)) + J_i, \quad i = 1, 2, \dots, n, \tag{1}$$

where $u_i(t)$ is the membrane potential of the ith neuron at the time t, $d_i > 0$ is a constant, $f_j(\cdot), g_j(\cdot) : \mathbb{R} \to \mathbb{R}$ represent the nonlinear activation functions of the jth neuron, a_{ij} and b_{ij} denote the synaptic connection weight of the jth neuron on the ith neuron at the time t and qt, respectively, T_{ijk} is the second-order synaptic weights of the neural networks, J_i is the external input, q is the proportional delay factor and satisfies $0 < q < 1$, $qt = t - (1-q)t$, in which $(1-q)t$ denotes the transmission delay and is the unbounded delay.

The initial conditions associated with the network (1) are of the form

$$u_i(s) = u_{i0}, \quad s \in [qt_0, t_0], \ i = 1, 2, \dots, n,$$

where t_0 is a constant. If $t_0 = 0$, $s \in [qt_0, t_0]$ is equal to $s = t_0 = 0$.

Let

$$u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T, \quad D = diag\{d_1, d_2, \dots, d_n\}, \quad A = (a_{ij})_{n\times n},$$

$$B = (b_{ij})_{n\times n}, \quad f(u(t)) = (f_1(u_1(t)), f_2(u_2(t)), \dots, f_n(u_n(t)))^T,$$

$$g(u(qt)) = (g_1(u_1(qt)), g_2(u_2(qt)), \cdots, g_n(u_n(qt)))^T,$$

$$G(u(qt)) = diag\{\underbrace{g(u(qt)), g(u(qt)), \dots, g(u(qt))}_{n}\},$$

$$T_i = (T_{ijk})_{n\times n}, \quad T = (T_1, \dots, T_n)^T, \quad J = (J_1, \dots, J_n)^T.$$

Then, the network (1) can be rewritten as follows:

$$\dot{u}(t) = -Du(t) + Af(u(t)) + Bg(u(qt)) + G^T(u(qt)) T g(u(qt)) + J. \tag{2}$$

Consider the transformation defined as

$$y_i(t) = u_i(e^t), \quad i = 1, 2, \dots, n, \tag{3}$$

Letting $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$, $\tau = -\ln q > 0$, then we have

$$\dot{y}(t) = e^t\{-Dy(t) + Af(y(t)) + Bg(y(t-\tau)) + G^T(y(t-\tau)) T g(y(t-\tau)) + J\}, \tag{4}$$

Thus, the initial condition associated with the network (4) is given by

$$y_i(s) = \varphi_i(s), \quad t_0 - \tau \le s \le t_0, \ i = 1, 2, \dots, n,$$

where $\varphi_i(s) \in C([t_0 - \tau, t_0], \mathbb{R})$ is a continuous function.

Remark 1. From the transformation (3), the system (2) is equivalent to the system (4). And the system (4) is a class of high-order neural networks with constant delay and variable coefficients. The system (4) is different from the models in [30–34]. The coefficients of the models in [30–34] are bounded time-invariant functions, but in this paper the coefficients containing e^t of the model (4) are unbounded time-varying functions.

In addition, the following definitions, lemmas and assumption are needed.

Definition 1. An equilibrium point $u^* = (u_1^*, u_2^*, \dots, u_n^*)^T$ of the system (2) is said to be pth ($p \in \{1, 2, \infty\}$) globally exponentially stable, if there exist two positive constants $M > 0$ and $\lambda > 0$ such that

$$\| u(t, t_0, u_0) - u^* \|_p \le M \| u_0 - u^* \|_p e^{-\lambda t}$$

holds, where $u_0 = (u_{10}, u_{20}, \dots, u_{n0})^T$ is the initial condition of the system (2), $u(t, t_0, u_0)$ is the solution of system (2).

Definition 2 (*Vidyasagar [35]*). For any real matrix $A = (a_{ij})_{n\times n}$, its matrix measure is defined as

$$\mu_p(A) = \lim_{\varepsilon \to 0^+} \frac{\| I + \varepsilon A \|_p - 1}{\varepsilon},$$

where $\| \cdot \|_p$ denotes the matrix norm in $\mathbb{R}^{n\times n}$, I is the identity matrix, $p \in \{1, 2, \infty\}$.

Let the matrix norm be as follows:

$$\| A \|_1 = \max_j \left\{ \sum_{i=1}^{n} |a_{ij}| \right\}; \quad \| A \|_2 = \sqrt{\lambda_{\max}(A^T A)}; \quad \| A \|_\infty = \max_i \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}.$$

Then, we can obtain the corresponding matrix measure as follows:

$$\mu_1(A) = \max_j \left\{ a_{jj} + \sum_{i=1, i\ne j}^{n} |a_{ij}| \right\}; \quad \mu_2(A) = \frac{1}{2}\lambda_{\max}(A^T + A); \quad \mu_\infty(A) = \max_i \left\{ a_{ii} + \sum_{j=1, j\ne i}^{n} |a_{ij}| \right\}.$$

Lemma 1 (*Vidyasagar [35]*). *From the definition of matrix measure, for any $A, B \in \mathbb{R}^{n\times n}$, $p = 1, 2, \infty$, we have*

(1) $-\| A \|_p \le \mu_p(A) \le \| A \|_p$;
(2) $\mu_p(\alpha A) = \alpha \mu_p(A)$, $\forall \alpha > 0$;
(3) $\mu_p(A + B) \le \mu_p(A) + \mu_p(B)$.

Lemma 2 (*Tian [36], A generalized Halanay's inequality*). *Suppose*

$$\dot{u}(t) \le \gamma(t) - \alpha(t) u(t) + \beta(t) \sup_{t-\tau \le \sigma \le t} u(\sigma)$$

holds for any $t \ge t_0$. Here $\tau \ge 0$, and $\gamma(t)$, $\alpha(t)$, $\beta(t)$ are continuous functions such that $0 \le \gamma(t) \le \gamma^*$, $\alpha(t) \ge \alpha_0$, $0 \le \beta(t) \le q\alpha(t)$ for any $t \ge t_0$ with constants $\gamma^* > 0$, $\alpha_0 > 0$, $0 \le q < 1$. Then we have

$$u(t) \le \frac{\gamma^*}{(1-q)\alpha_0} + G e^{-\mu^*(t-t_0)}$$