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A new condition for robust stability of uncertain neural networks with time delays

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ABSTRACT

This paper is concerned with the global asymptotic stability problem of dynamical neural networks with multiple time delays under parameter uncertainties. First carrying out an analysis of existence and uniqueness of the equilibrium point by means of the Homeomorphism theory, and then, studying the global asymptotic stability of the equilibrium point by constructing a suitable Lyapunov functional, we derive a new global robust stability criterion for the class of delayed neural networks with respect to the Lipschitz activation functions. The result obtained establishes a relationship between the neural network parameters only and it is independent of the time delay parameters. It is shown that the established stability condition generalizes some existing ones and it can be considered to an alternative result to some other corresponding results derived in previous literature. We also give some comparative numerical examples to demonstrate the validity and effectiveness of our proposed result.

1. Introduction

In recent years, neural networks of various classes such as bidirectional associative memory neural networks, cellular neural networks, Cohen–Grossberg neural networks and Hopfield neural networks have been paid a great deal of attention due to their potential applications to many engineering problems such as combinatorial optimization, adaptive control, signal processing, associative memories and pattern recognition. In many applications of neural networks to practical engineering problems, the information to be processed is in the form of stable states. Therefore, in order to successfully employ neural networks in these problems, we need to know the stability and equilibrium properties of the designed neural network. Specially, in order to carry out a precise stability analysis of electronically implemented neural networks, some key parameters must be considered when forming the system equations of neural networks. First, some delay parameters, occurring during the process of information processing, must be introduced into the system equations of neural networks. Second, we must take into account the uncertainties in the network parameters, caused by some external disturbances. Accordingly, various important and interesting results have been reported on different classes of neural networks with time delays and parameter uncertainties based on the different approaches [1–44].

One of the model of neural networks with time delays is described by the following sets of differential equations:

$$\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j)) + u_i, i = 1, 2, \ldots, n \tag{1}$$

which can be written in the vector-matrix form as follows:

$$\frac{dx(t)}{dt} = -Cx(t) + Af(x(t)) + Bf(x(t-\tau)) + u$$

where n is the number of the neurons, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n$ denotes the state vector of the neurons with $x_i(t)$ denoting the state of the neuron i at time t, $C = diag(c_i) > 0$ is a positive diagonal matrix, $A = (a_{ij})$ and $B = (b_{ij})$ are the interconnection weight matrix and the delayed interconnection weight matrix, respectively. $f(x(t)) = (f(x_1(t)), f(x_2(t)), \ldots, f(x_n(t)))^T \in R^n$ denotes the neuron activations, τ_j denotes the time delay associated with the jth neuron, $f(x(t-\tau)) = (f(x_1(t-\tau_1)), f(x_2(t-\tau_2)), \ldots, f(x_n(t-\tau_n)))^T \in R^n$, $u = (u_1, u_2, \ldots, u_n)^T \in R^n$ is a constant input vector.

Note that, in neural network model (1), the time delay required in transmitting a signal from the neuron j to all neurons is assumed to be the same and denoted by τ_j. In this case, the number of the delay parameters is equal to n. However, it will be more practical and realistic to denote the delay parameters by τ_{ij} which is the transmission delay from the neuron j to the neuron i. In this case, the number of the delay parameters is equal to n^2, and

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neural network model (1) takes the following form:

$$\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + u_i, \quad i = 1, 2, \dots, n \tag{2}$$

The uncertainties in the network parameters $A = (a_{ij})$, $B = (b_{ij})$ and $C = diag(c_i) > 0$ are assumed to be formulated as follows:

$$\begin{aligned} C_I &:= \{C = diag(c_i) : 0 < \underline{C} \le C \le \overline{C},\ i.e., 0 < \underline{c}_i \le c_i \le \overline{c}_i, \forall i\} \\ A_I &:= \{A = (a_{ij}) : \underline{A} \le A \le \overline{A},\ i.e., \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, \quad i,j = 1, 2, \dots, n\} \\ B_I &:= \{B = (b_{ij}) : \underline{B} \le B \le \overline{B},\ i.e., \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij}, \quad i,j = 1, 2, \dots, n\} \end{aligned} \tag{3}$$

We will also assume that the neuron activation functions f_i are Lipschitz continuous satisfying

$$|f_i(x) - f_i(y)| \le \ell_i |x - y|, \quad i = 1, 2, \dots, n, \quad \forall x, y \in R, x \ne y$$

where $\ell_i > 0$ denotes a Lipschitz constant. This class of functions is denoted by $f \in \mathcal{L}$.

Recently, many authors studied robust stability of neural network model (1) by employing various types of Lyapunov–Krasovskii functionals and using the linear matrix inequality (LMI) approach [2,3,22,30,34–37]. Since the neural network model (1) can be written in the vector-matrix form, the LMI is an appropriate technique for the stability analysis of this class of delayed neural networks as the obtained stability conditions can be expressed in the LMI forms, and then the validity of the conditions can be easily checked by resorting to the Matlab LMI toolbox. On the other hand, since the neural network model (2) cannot be written in the vector-matrix form, the LMI is not an appropriate technique and approach for the stability analysis of this class of delayed neural networks. (To the best of our knowledge, no results in the LMI forms for the neural network model (2) have been reported in the literature). Therefore, in order to derive sufficient conditions for robust stability of neural system (2), one needs to develop different techniques and approaches. In the current literature, only few papers studied the robust stability of neural system (2) in which the stability results are derived by employing the M-matrix condition [18–21,31] and using some matrix-norm inequalities [6–9,24–27].

In this paper, by constructing a suitable Lyapunov functional and using the properties some certain matrix-norm inequalities, we derive a new sufficient condition for the existence, uniqueness and global asymptotic stability of the equilibrium for the neural network model (2) under parameter uncertainties given by (3) and with respect to the Lipschitz activation functions.

Throughout this paper, we will use the notations for vectors and matrices: For the vector $v = (v_1, v_2, \dots, v_n)^T$, $|v|$ will denote $|v| = (|v_1|, |v_2|, \dots, |v_n|)^T$. For any real matrix $Q = (q_{ij})_{n \times n}$, $|Q|$ will denote $|Q| = (|q_{ij}|)_{n \times n}$. For the vector v and the matrix Q, we note the following three commonly used vector and matrix norms:

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|, \quad \|v\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2}, \quad \|v\|_\infty = \max_{1 \le i \le n} |v_i|$$

and

$$\|Q\|_1 = \max_{1 \le i \le n} \sum_{j=1}^{n} |q_{ji}|$$

$$\|Q\|_2 = [\lambda_{\max}(Q^T Q)]^{1/2}$$

$$\|Q\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |q_{ij}|$$

Before we proceed presenting our main results, we will need to give some previous literature lemmas which define upper bound norms for the interval matrices, that are very important in the context of this paper.

Lemma 1 (*Faydasicok and Arik [9]*). *Let A be any real matrix defined by $A \in A_I := \{A = (a_{ij}) : \underline{A} \le A \le \overline{A},$ i.e., $\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, i,j = 1, 2, \dots, n\}$. Define $A^* = \frac{1}{2}(\overline{A} + \underline{A})$ and $A_* = \frac{1}{2}(\overline{A} - \underline{A})$. Let*

$$\sigma_1(A) = \sqrt{\| |A^{*T} A^*| + 2|A^{*T}| A_* + A_*^T A_* \|_2}$$

Then, the following inequality holds

$$\|A\|_2 \le \sigma_1(A)$$

Lemma 2 (*Cao et al. [3]*). *Let A be any real matrix defined by $A \in A_I := \{A = (a_{ij}) : \underline{A} \le A \le \overline{A},$ i.e., $\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, i,j = 1, 2, \dots, n\}$. Define $A^* = \frac{1}{2}(\overline{A} + \underline{A})$ and $A_* = \frac{1}{2}(\overline{A} - \underline{A})$. Let*

$$\sigma_2(A) = \|A^*\|_2 + \|A_*\|_2$$

Then, the following inequality holds

$$\|A\|_2 \le \sigma_2(A)$$

Lemma 3 (*Ensari and Arik [6]*). *Let A be any real matrix defined by $A \in A_I := \{A = (a_{ij}) : \underline{A} \le A \le \overline{A},$ i.e., $\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, i,j = 1, 2, \dots, n\}$. Define $A^* = \frac{1}{2}(\overline{A} + \underline{A})$ and $A_* = \frac{1}{2}(\overline{A} - \underline{A})$. Let*

$$\sigma_3(A) = \sqrt{\|A^*\|_2^2 + \|A_*\|_2^2 + 2\|A_*^T |A^*|\|_2}$$

Then, the following inequality holds

$$\|A\|_2 \le \sigma_3(A)$$

Lemma 4 (*Singh [30]*). *Let A be any real matrix defined by $A \in A_I := \{A = (a_{ij}) : \underline{A} \le A \le \overline{A},$ i.e., $\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, i,j = 1, 2, \dots, n\}$. Define $\hat{A} = (\hat{a}_{ij})_{n \times n}$ with $\hat{a}_{ij} = \max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\}$. Let*

$$\sigma_4(A) = \|\hat{A}\|_2$$

Then, the following inequality holds

$$\|A\|_2 \le \sigma_4(A)$$

2. Existence and uniqueness analysis

In this section, we will present the following theorem that states a new sufficient condition for the existence and uniqueness of the equilibrium point of the neural network model (2):

Theorem 1. *For the neural network defined by (2), assume that the network parameters satisfy (3) and $f \in \mathcal{L}$. Then, the neural network model (2) has a unique equilibrium point for every input vector u if the following condition holds*

$$\psi = c_m - \ell_M \sigma_m(A) - \sqrt{n} \ell_M \mu_m(B) > 0$$

where $\sigma_m(A) = \min\{\sigma_1(A), \sigma_2(A), \sigma_3(A), \sigma_4(A)\}$, $c_m = \min\{\underline{c}_i\}$, $\ell_M = \max\{\ell_i\}$, $\mu_m(B) = \min\{\sqrt{\|R\|_\infty}, \sqrt{\|R\|_1}\}$, $R = (r_{ij})_{n \times n}$ with $r_{ij} = \hat{b}_{ij}^2$ and $\hat{b}_{ij} = \max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\}$.

Proof. In order to show the existence of a unique equilibrium point of system (2) for every u, we will employ the homeomorphism mapping theorem. To this end, we first define following mapping associated with system (2):

$$H(x) = -Cx + Af(x) + Bf(x) + u \tag{4}$$

If $H(x) \ne H(y)$ for all $x \ne y$ and $\|H(x)\| \to \infty$ as $\|x\| \to \infty$ then, we will say that $H(x)$ is homeomorphism of R^n. It is well known that if $H(x)$ is homeomorphism of R^n, then $H(x) = 0$ has a unique solution for every u. Now, if x^* is an equilibrium point of neural network model (2), by definition, it satisfies

$$-Cx^* + Af(x^*) + Bf(x^*) + u = 0$$

Note that the solution of $H(x) = 0$ is an equilibrium point of (2). Therefore, in order to ensure the existence and uniqueness of the equilibrium point of neural system (2), it will be sufficient to show