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Global dissipativity of fractional-order neural networks with time delays and discontinuous activations

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ABSTRACT

In this paper, we investigate global dissipativity of a class of fractional-order neural networks (FNNs) with time delays and discontinuous activations. The relevant results are in the sense of Caputo's fractional derivation. The existence of global solutions in Filippov's sense can be guaranteed by using nonsmooth analysis, differential inclusion theory and the properties of fractional calculus. Based on comparison theorem and stability theorem for a class of fractional-order systems with time delays, some new sufficient conditions are derived to ensure dissipativity of solutions, meanwhile, the globally attractive set is given. The obtained results enrich and enhance the earlier reports. Finally, two numerical examples are given to demonstrate the effectiveness of the obtained results.

1. Introduction

Fractional-order neural networks (FNNs), as a kind of important biological networks, have received considerable attentions [1–6]. On the one hand, fractional derivatives have non-locality and weak singularity. More importantly, compared with the classical integer-order derivatives, fractional-order derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [7–10]. On the other hand, FNNs have infinite memory, and fractional-order parameters can enrich the system performance by increasing one degree of freedom [11–13]. It is generally known that time delays are unavoidable in hardware implementation due to finite switching speed of the amplifiers and communication time. In addition, the existence of time delays may lead to some complex dynamic behaviors such as oscillation, divergence, chaos, instability, or other poor performance of the neural networks. Therefore, it is valuable and practical to investigate FNNs with time delays.

To the best of our knowledge, most results of FNNs were established on the premise of Lipschitz-continuous activations. In fact, discontinuous activations have been proved really useful as an ideal model of activations with very-high gain, and such models have been frequently applied to solve constrained optimization problems via a sliding mode approach [14,15]. What is more, Forti and Nistri have pointed out that neural networks with discontinuous activations were frequently encountered in the applications in [16], such as impacting machines, systems oscillating under the effect of an earthquake and dry friction [17–19]. Subsequently, more and more researchers pay more attentions to studying neural networks with discontinuous activations, such as [17,20–24]. Nevertheless, these results were built in the case of integer-order cases. As the well-studied integer-order systems are the special cases of the fractional-order systems, and the fractional-order models can describe the systems more precise than the integer-order models in practice. Therefore, it is necessary to consider discontinuous activations in the dynamic analysis of FNNs.

Dissipativity, which is introduced in the early 1970s, is an important property of dynamical systems. The concept of dissipativity in dynamical systems is more general and generalizes the idea of a Lyapunov function. In addition, it has found applications in the areas such as stability theory, chaos and synchronization theory, system norm estimation, and robust control [25,26]. Although dissipativity analysis has long been studied in theory and applications of integer-order neural networks [27–32], few authors have discussed the dissipativity of FNNs.

Based on above motivations, in this paper, we introduce a class of FNNs with time delays and discontinuous activations, and investigate the global dissipativity of such system. More precisely, the contributions of this paper are described below:

(1) Owing to the discontinuities of activations, we introduce the concept of Filippov solution in the sense of Caputo's fractional derivation. Furthermore, based on nonsmooth analysis and differential inclusion theory, we investigate the existence of global

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solution in the sense of Filippov for FNNs with time delays and discontinuous activations.

(2) By employing comparison theorem and stability theorem for a class of fractional-order systems with time delays, some sufficient criteria for global dissipativity of FNNs with time delays and discontinuous activations are presented.

(3) Most results of FNNs have not considered time delays and discontinuous activations, but our results make it up.

(4) Compared with the previous results, the well-studied integer-order neural networks are the special cases of the FNNs. Moreover, we extend the results of fractional-order delayed neural networks with continuous activations [33]. So the results in this paper are less conservative and more general.

The organization of this paper is as follows. The systems and some preliminaries are introduced in Section 2. In Section 3, sufficient criteria are established by using the theory of fractional-order differential equations with discontinuous right-hand sides, comparison theorem and stability theorem for a class of fractional-order systems with time delays. Then, numerical simulations are given to demonstrate the effectiveness of the obtained results in Section 4. Finally, conclusions are drawn in Section 5.

2. Preliminaries and system description

Notations: Through this paper, $\mathbb{R}$ is the space of real number, $\mathbb{R}_+$ denotes the set of all nonnegative real numbers, $\mathbb{N}_+$ is the set of positive integers, and $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $[\cdot,\cdot]$ represents the interval. a.a. implies almost all. If $x \in \mathbb{R}^n$, we have $\|x\|_1 = \sum_{i=1}^{n} |x_i|$. If $F : E \hookrightarrow \mathbb{R}^n$ $(E \subset \mathbb{R}^n)$ is a set-valued map, we have $\|F(x)\| = \sup_{\gamma \in F(x)} \|\gamma\|_1$. In addition, $C^r([t_0, +\infty), \mathbb{R})$ denotes the space consisting of r-order continuous differentiable functions from $[t_0, +\infty)$ into $\mathbb{R}$.

In this section, we report a number of definitions and properties concerning fractional calculation, which are needed in the development. In addition, some useful lemmas are presented.

2.1. Caputo fractional-order derivative

Definition 1 (*Kilbas et al.* [34]). The fractional-order integral of order α for an integrable function $f(t) : [t_0, +\infty) \to \mathbb{R}$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau,$$

where $\alpha > 0$, and $\Gamma(\cdot)$ is the Gamma function which is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad (Re(z) > 0).$$

where $Re(z)$ is the real part of z.

Definition 2 (*Kilbas et al.* [34]). The Caputo fractional-order derivative of order α for a function $f(t) \in C^{n+1}([t_0, +\infty), \mathbb{R})$ is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau,$$

where $t \ge t_0$ and n is a positive integer such that $n-1 < \alpha < n$. Particularly, when $0 < \alpha < 1$,

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}}\, d\tau.$$

The Laplace transform of the Caputo fractional-order derivative is

$$\mathcal{L}\{D^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(t_0),$$

where $n-1 < \alpha \le n$, $F(s)$ is the Laplace transform of $f(t)$ with $F(s) = \mathcal{L}\{f(t)\}$, and s is the variable in Laplace domain.

In addition, the following properties about Caputo fractional-order derivative and a necessary Lemma are given.

Property 1. *$D^\alpha c = 0$ holds, where c is any constant.*

Property 2. *For any constants ν_1 and ν_2, the linearity of Caputo fractional-order derivative is described by*

$D^\alpha(\nu_1 f(t) + \nu_2 g(t)) = \nu_1 D^\alpha f(t) + \nu_2 D^\alpha g(t)$.

Lemma 1 (*Kilbas et al.* [34]). *Let $\Omega = [a, b]$ be an interval on the real axis $\mathbb{R}$, let $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}_+$ or $n = \alpha$ for $\alpha \in \mathbb{N}_+$. If $x(t) \in C^n[a, b]$, then*

$$I^\alpha D^\alpha x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!} (t-a)^k, \quad n-1 < \alpha \le n,$$

where I^α is the fractional-order integral of order α and D^α is the Caputo fractional-order derivative of order α. In particular, if $0 < \alpha \le 1$ and $x(t) \in C^1[a, b]$, then

$I^\alpha D^\alpha x(t) = x(t) - x(a)$.

2.2. System description

We consider a class of FNNs with time delays and discontinuous activations described by the following equation:

$$D^\alpha x_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-h)) + I_i(t), \tag{1}$$

where $i = 1, 2, \ldots, n (n \in \mathbb{N}_+)$, $t \ge 0$, D^α denotes the Caputo fractional derivative of order α and $0 < \alpha < 1$; $x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the vector of neuron states; $d_i > 0$ is a constant; a_{ij} and b_{ij} are constants which represent the neuron interconnection weight and the delayed neuron interconnection weight, respectively; and $I_i(t)$ is a continuous bounded external input function. $x_i(s) = \varphi_i(s) \in C([-h, 0], \mathbb{R})$ is the initial condition of system (1) where $C([-h, 0], \mathbb{R})$ is Banach space of all continuous functions and time delay $h > 0$. Moreover, $f(x) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))^T$ and $g(x) = (g_1(x_1), g_2(x_2), \ldots, g_n(x_n))^T$ are vector-valued activation functions from $\mathbb{R}^n$ to $\mathbb{R}^n$.

The following assumption about activation functions f_j and g_j for $j = 1, 2, \ldots, n$ is given for system (1).

(H1) The activation functions $f_j \in \mathcal{F}$ (and $g_j \in \mathcal{G}$) for any $j = 1, 2, \ldots, n$, where $\mathcal{F}$ (and $\mathcal{G}$) denotes the class of functions from $\mathbb{R}$ to $\mathbb{R}$ which are continuous and have at most a finite number of jump discontinuities ρ_k (and ϱ_k) in every bounded interval. In addition, there exist finite right and left limits, $f_j(\rho_k^+)$ (and $g_j(\varrho_k^+)$) and $f_j(\rho_k^-)$ (and $g_j(\varrho_k^-)$) respectively.

Remark 1. In the previous results of FNNs [33,35–38], the nonlinear activation functions f and g satisfy the common Lipschitz conditions, but in this paper these conditions are removed. In addition, compared with the results of integer-order neural networks with discontinuous activations, the boundedness [39,40] and the monotonicity [39–42] of the activation functions are not required. So our assumption are more general.

Due to the presence of discontinuous activations f_j and g_j $(j = 1, 2, \ldots, n)$, system (1) is discontinuous and its classical solution does not exist. Here, we consider the solutions of system under the framework of Filippov. Now, the concept of Filippov solution [43] is given.

Definition 3 (*Filippov* [43]). For a system with discontinuous right-hand sides:

$$\frac{dx}{dt} = f(t, x_t), \quad t \ge 0, \tag{2}$$