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Doubly periodic traveling waves in cellular neural networks with polynomial reactions

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ABSTRACT

In a study (Szekely, 1965) [1] of the locomotion of salamanders, it is observed that a 'doubly periodic traveling wave solution' of a logical neural network can be used to explain a dynamic pattern of movements. We show here that nonlinear and nonlogical artificial neural network can also be built by means of reaction diffusion principles and existence or nonexistence of doubly periodic traveling waves can be guaranteed by adjusting parameters built into these networks. Our derivations for existence are based on implicit function theorems and the invariance properties of our model; while nonexistence is based on boundedness properties of the polynomial reaction term. We also give illustrative examples as well as comments on the differences between present results with those obtained for linear models studied earlier in Cheng and Lin (2009) [2,3].

1. Introduction

Szekely in [1] studied the movement of salamanders and found that the dynamic locomotive pattern of salamanders can be simulated by periodic (vector) sequences $\{v^{(t)}\}_{t=0}^{\infty}$, where each $v^{(t)}$ is also a periodic real sequence of the form $\{\dots,v_0^{(t)},v_1^{(t)},\dots\}$ and these sequences can be generated by a bipolar neural network. In [2,3], the authors observed that these vector sequences have several characteristic properties, namely, they satisfy

$$v_i^{(t+\tau)} = v_{i+\delta}^{(t)} \quad \text{(temporal}-\text{spatial transition condition)},$$

$$v_i^{(t)} = v_{i+\Upsilon}^{(t)} \quad \text{(spatial periodicity condition)},$$

and

$$v_i^{(t+\Delta)} = v_i^{(t)} \quad \text{(temporal periodicity condition)},$$

where $t \in \{0,1,\dots,\}$, τ,Δ,Υ are positive integers and δ,i are integers. Such vector sequences are called *doubly periodic* (Δ,Υ)–traveling waves and can also be used to explain other dynamic patterns of movements (see e.g. [3]).

A natural question then is whether we can build artificial neural networks which can yield such traveling waves. To this end, in [2,3], we apply reaction and diffusion principles to build several dynamic network models and obtain the exact conditions the required traveling waves may or may not be generated by them. The network in [2] has a linear 'diffusion part' and a nonlinear 'reaction part'. However, the reaction part consists of a quadratic polynomial so the investigation is reduced to a linear homogeneous problem. The network in [3] has the same diffusion part, but has a linear 'reaction part'. It is therefore of great interests to build *nonlinear* networks with general polynomials as the reaction terms, and see if the desired traveling waves exist (for readers who are more interested in the design problem, the material presented in the last part of this paper can be consulted first).

We first briefly recall the basic reaction and diffusion principles in [2,3] for building our networks. Let $\mathbf{R}=(-\infty,\infty)$, $\mathbf{N}=\{0,1,\dots\}$, $\mathbf{Z}=\{\dots,-1,0,1,\dots\}$ and $\mathbf{Z}^+=\{1,2,\dots\}$. Let v_i be the i-th neuron pool, where $i \in \mathbf{Z}$, and $v_i^{(t)}$ be the state value of v_i in the time period $t \in \mathbf{N}$. Suppose all v_i are placed in an infinite grid such that v_{i-1} and v_{i+1} are the left and right neighbors of v_i respectively. Additionally, we assume that for any neuron pool v_i, the transmission of information from time t to time $t+1$ is affected by itself and its neighbors, and that there is a control mechanism imposed. Then, it is reasonable to consider the equation

$$v_i^{(t+1)} - v_i^{(t)} = \alpha(v_{i-1}^{(t)} - 2v_i^{(t)} + v_{i+1}^{(t)}) + g(v_i^{(t)}), \quad i \in \mathbf{Z}; \quad t \in \mathbf{N}, \tag{1}$$

where α is treated as a fixed real proportionality constant and g is a real (control) function.

In [2], we have studied thoroughly the cases $g(x)=\kappa x^2$ or $g(x)=0$ for $x \in \mathbf{R}$; and in [3], the cases $g(x)=\kappa$ or $g(x)=\kappa(x-\lambda)$, where κ,λ are real parameters.

In this paper, we will assume that g is a polynomial with simple roots modulo the cases just described. More specifically,

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we assume throughout that Λ is the *ordered* set

$$\Lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_s\}, \tag{2}$$

where $s\geq 2$ and $\lambda_1<\lambda_2<\cdots<\lambda_s$. We will also assume throughout the rest of our paper that g in (1) satisfies

$$g(x)=\kappa f(x), \quad x\in\mathbf{R}; \quad \kappa\in\mathbf{R}\backslash\{0\}, \tag{3}$$

where

$$f(x)=(x-\lambda_1)\cdots(x-\lambda_s), \quad x\in\mathbf{R}, \tag{4}$$

i.e., all roots of f are simple and belong to Λ.

We remark that under the assumption (4), the well-known discrete Nagumo equation

$$v_i^{(t+1)}-v_i^{(t)}=\alpha(v_{i-1}^{(t)}-2v_i^{(t)}+v_{i+1}^{(t)})+\kappa v_i^{(t)}(v_i^{(t)}-a)(1-v_i^{(t)}), \quad i\in\mathbf{Z}; \ t\in\mathbf{N}, \tag{5}$$

where $a\in(0,1)$, can be regarded as a special case of (1). Eq. (5) has been studied in many papers such as [4,5]. Additionally, the discrete Fisher equation

$$v_i^{(t+1)}-v_i^{(t)}=\alpha(v_{i-1}^{(t)}-2v_i^{(t)}+v_{i+1}^{(t)})+\kappa v_i^{(t)}(1-v_i^{(t)}), \quad i\in\mathbf{Z}; \quad t\in\mathbf{N} \tag{6}$$

is also a special case of (1) which is considered in [6].

Our main concern is the existence and nonexistence of doubly periodic traveling wave solutions of (1) where g is given by (3). We will be able to show that when $|\kappa|$ is sufficiently large, such doubly periodic traveling wave solutions exist, while when $|\kappa|$ is sufficiently small (but $\kappa\neq 0$ by assumption), some doubly periodic traveling wave solutions cannot exist.

To this end, we first formalize in the next section the definition of 'doubly periodic traveling wave', then we give the concept of '$\omega-$traveling wave' and show that seeking 'doubly periodic traveling wave solutions' is equivalent to seeking '$\omega-$traveling wave solutions'. In Section 3, we provide several preliminary lemmas. In Section 4, we further study the properties of '$\omega-$traveling wave' solutions such as their distributions, boundedness, etc. In Section 5, we will show the existence of '$\omega-$periodic traveling wave' solutions for our model under appropriate assumptions. In Section 6, we will provide nonexistence of '$\omega-$traveling wave' solutions under appropriate assumptions. In the final section, we supply several examples to illustrate our conclusions in the previous sections.

2. ω-Traveling wave solutions

First, recall that a real double sequence $\{v_i^{(t)}\}_{t\in\mathbf{N},i\in\mathbf{Z}}$ is called a traveling wave with velocity $-\delta/\tau$ if $v_i^{(t+\tau)}=v_{i+\delta}^{(t)}$ for all $i\in\mathbf{Z}$ and $t\in\mathbf{N}$, where $\tau\in\mathbf{Z}^+$ and $\delta\in\mathbf{Z}$. Recall also that given a sequence $\varphi=\{\varphi_m\}$, if $\omega\in\mathbf{Z}^+$ such that $\varphi_{m+\omega}=\varphi_m$ for any $m\in\mathbf{Z}$, then ω is called a period of φ; furthermore, if ω is the *least* among all periods of φ, then φ is said to be $\omega-$periodic. A simple result about periodic sequence is the following.

Lemma 2.1 (*Cheng and Lin [2]*). *If* $\mathbf{y}=\{y_i\}$ *is* $\omega-$*periodic and* ω_1 *is a period of* $\mathbf{y}$, *then* $\omega \bmod \omega_1=0$.

Suppose $\mathbf{v}=\{v_i^{(t)}\}_{t\in\mathbf{N},i\in\mathbf{Z}}$ is a double sequence. A positive number ξ is called a spatial period of $\mathbf{v}$ if $v_{i+\xi}^{(t)}=v_i^{(t)}$ for all i and t; furthermore, if ξ is the least among all spatial periods of $\mathbf{v}$, then $\mathbf{v}$ is called spatial $\xi-$periodic. Similarly, a positive number η is called a temporal period of $\mathbf{v}$ if $v_i^{(t+\eta)}=v_i^{(t)}$ for all i and t and if η is the least among all temporal periods of $\mathbf{v}$, then $\mathbf{v}$ is called temporal $\eta-$periodic. Let $\tau\in\mathbf{Z}^+$ and $\delta\in\mathbf{Z}$. A double sequence $\mathbf{v}$ is said to be a doubly periodic traveling wave with velocity $-\delta/\tau$ if it is a traveling wave with velocity $-\delta/\tau$ and it also has spatial and temporal periods. If $\mathbf{v}$ is temporal $\Delta-$periodic, spatial $\Upsilon-$periodic and is a traveling wave with velocity $-\delta/\tau$, then $\mathbf{v}$ is called a $(\Delta,\Upsilon)-$periodic traveling wave with velocity $-\delta/\tau$.

A double sequence $\mathbf{v}=\{v_i^{(t)}\}_{t\in\mathbf{N},i\in\mathbf{Z}}$ is called a solution of (1) if it renders (1) into an identity after substitution. In this paper, we are mainly concerned with the existence and nonexistence of $(\Delta,\Upsilon)-$periodic traveling wave solutions of (1) with velocity $-\delta/\tau$. Such solutions possess basic properties which will be recalled as follows. Their proofs can be found in [2] or can be obtained by slightly modifying the corresponding proofs in [2].

Lemma 2.2 (*cf. Cheng and Lin [2, Proof of Theorem 2]*). *Let* $\tau\in\mathbf{Z}^+$ *and* $\delta\in\mathbf{Z}$. *If* $\{v_i^{(t)}\}$ *is a traveling wave solution of* (1) *with velocity* $-\delta/\tau$, *then* $\{w_i^{(t)}\}=\{v_{-i}^{(t)}\}$ *is also a traveling wave solution of* (1) *with velocity* δ/τ.

In view of Lemma 2.2, we may restrict our attention to traveling wave solutions with velocity $-\delta/\tau$ for $\delta\in\mathbf{N}$ and $\tau\in\mathbf{Z}^+$. Note that a traveling wave with velocity $-\delta/\tau$ is also a traveling wave with velocity $-(k\delta)/(k\tau)$, where $\delta\in\mathbf{N}$ and $\tau,k\in\mathbf{Z}^+$. For this reason, in the sequel, we will only consider τ and δ that satisfy $\tau\in\mathbf{Z}^+$ and $\delta\in\mathbf{N}$ such that their greatest common divisor $(\tau,\delta)_*=1$. Such a pair of relatively prime integers and the corresponding velocity $-\delta/\tau$ are said to be *admissible*.

Lemma 2.3 (*cf. Cheng and Lin [2, Proof of Theorem 2]*). *Suppose* (τ,δ) *is admissible. If* $\{v_i^{(t)}\}$ *is a traveling wave solution of* (1) *with velocity* $-\delta/\tau$, *then the sequence* $\mathbf{y}=\{y_m\}_{m\in\mathbf{Z}}$ *defined by*

$$y_{\tau i+\delta t}=v_i^{(t)}, \quad i\in\mathbf{Z}; \ t\in\mathbf{N} \tag{7}$$

is well defined on $\mathbf{Z}$ *and is a solution of the equation*

$$\varphi_{m+\delta}-\varphi_m=\alpha(\varphi_{m-\tau}-2\varphi_m+\varphi_{m+\tau})+\kappa f(\varphi_m), \quad m\in\mathbf{Z}, \tag{8}$$

where $\kappa\in\mathbf{R}\backslash\{0\}$, $\alpha\in\mathbf{R}$ *and* f *is the function defined by* (4). *Conversely, if* $\mathbf{y}=\{y_m\}$ *is a solution of* (8), *then the double sequence* $\{v_i^{(t)}\}$ *defined by* (7) *is a traveling wave solution of* (1) *with velocity* $-\delta/\tau$.

Suppose (τ,δ) is admissible. In view of Lemma 2.3, we see that there is a one-to-one correspondence between all traveling wave solutions of (1) with velocity $-\delta/\tau$ and all solutions of (8). A double sequence $\{v_i^{(t)}\}$ is called an ω-*traveling wave solution*[1] of (1) with velocity $-\delta/\tau$ if $\{v_i^{(t)}\}$ is a traveling wave solution of (1) with velocity $-\delta/\tau$ and the '*associated sequence*' $\mathbf{y}=\{y_m\}$ defined by (7) is $\omega-$periodic.

In the following result, we will need the concepts of (positive) integral complements and least (positive) integral multiples of positive fractions. More specifically, given positive integers α and β, any positive integer ψ' such that $\psi'\alpha/\beta$ is also an integer is called an integral complement of α/β; and least integral complements of α/β is denoted by $\psi(\alpha,\beta)$. The number $\psi(\alpha,\beta)\alpha/\beta$ is called the least integral multiple of α/β.

Lemma 2.4 (*cf. Cheng and Lin [2, Proof of Theorem 4]*). *Suppose* (τ,δ) *is admissible with* $\delta\neq 0$. *If* $\mathbf{v}$ *is an* ω-*traveling wave of* (1) *with velocity* $-\delta/\tau$, *then* $\mathbf{v}$ *is also a* (Δ,Υ)-*traveling wave solution of* (1) *with velocity* $-\delta/\tau$, *where* Υ *and* Δ *are the least positive integral multiples of* ω/τ *and* ω/δ *respectively. Conversely, if* $\mathbf{v}$ *is a* (Δ,Υ)-*periodic traveling wave solution of* (1) *with velocity* $-\delta/\tau$, *then* $\mathbf{v}$ *is an* ω-*traveling wave with velocity* $-\delta/\tau$, *where* $\omega=(\tau\Upsilon,\delta\Delta)_*$, *furthermore,* Υ *and* Δ *are the least integral multiples of* ω/τ *and* ω/δ *respectively.*

For the case where (τ,δ) is admissible with $\delta=0$. We also have a similar result.

Lemma 2.5. *Suppose* (τ,δ) *is admissible with* $\delta=0$. *If* $\mathbf{v}$ *is a* ω-*traveling wave solution of* (1) *with velocity* $-\delta/\tau$, *then* $\mathbf{v}$ *is also a* $(\omega,1)$-*traveling wave solution of* (1) *with velocity* $-\delta/\tau$. *Conversely, if* $\mathbf{v}$ *is a*

[1] Also called $\omega-$periodic traveling wave solution in [2].