Neurocomputing 74 (2011) 689–707

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Stability and bifurcation analysis of a six-neuron BAM neural network model with discrete delays ☆

Changjin Xu [a,b], Xianhua Tang [b,*], Maoxin Liao [b,c]

[a] School of Mathematics and Statistics,Guizhou Key Laboratory of Economics System Simulation, Guizhou College of Finance and Economics, Guiyang, Guizhou 550004, PR China
[b] School of Mathematical Science and Computing Technology, Central South University, Changsha, Hunan 410083, PR China
[c] School of Mathematics and Physics, Nanhua University, Hengyang 421001, PR China

ARTICLE INFO

Article history:
Received 17 September 2009
Received in revised form
25 July 2010
Accepted 13 September 2010
Communicated by N. Ozcan
Available online 28 October 2010

Keywords:
Neural network
Stability
Hopf bifurcation
Discrete delay
Periodic solution

ABSTRACT

In this paper, a six-neuron BAM neural network model with discrete delays is considered. By analyzing the associated characteristic transcendental equation, the linear stability of the model is investigated and Hopf bifurcation is demonstrated. Some explicit formulae determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations are obtained by using the normal form method and center manifold theory. Finally, numerical simulations supporting the theoretical analysis are given.

1. Introduction

Recently, the dynamics properties (including stable, unstable, oscillatory and chaotic behavior) of neural networks with delays have attracted great attention of many researchers because of the successful application of neural networks to many fields such as intelligent control, optimization solvers, associative memories (or pattern recognition) etc., and many excellent and interesting results have been obtained (see [2–4,7,17–22]). It is well known that neural networks are complex and large-scale nonlinear dynamical systems, while the dynamics of the delayed neural networks are even rich and more complicated [6]. In order to obtain a deep and clear understanding of the dynamics of neural networks, many researchers have focused on the studying of simple systems. One of usual ways is to investigate the delayed neural network models with two, three or four neurons, see [2,5,11,12–14,16]. It is expected that we can gain some light for our understanding about the large networks by discussing the dynamics of two, three or four neurons networks (see [8–10] and the references cited therein). But there are inevitably some complicated problems if the simplified networks are carried over to large-scale networks, for example, the characteristic equation and the bifurcating periodic solutions are very complicated. So it is necessary to investigate the large-scale neural networks themselves. It is known to all that the delayed bidirectional associative memory neural network is described by the following system:

$$\begin{cases} \dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{m} c_{ji} f_i(y_j(t-\tau_{ji})) + I_i, & i = 1,2,\ldots,n, \\ \dot{y}_j(t) = -v_j y_j(t) + \sum_{i=1}^{n} d_{ij} g_j(x_i(t-v_{ij})) + J_j, & j = 1,2,\ldots,m, \end{cases} \tag{1.1}$$

where c_{ji}, d_{ij} ($i=1,2,\ldots,n$; $j=1,2,\ldots,m$) are the connection weights through neurons in two layers: the *I*-layer and *J*-layer; μ_i and v_j describe the stability of internal neuron processes on the *I*-layer and *J*-layer, respectively. On the *I*-layer, the neurons whose states are denoted by $x_i(t)$ receive the inputs I_i and the inputs outputted by those neurons in the *J*-layer via activation functions f_i, while on the *J*-layer, the neurons whose associated states are denoted by $y_j(t)$ receive the inputs J_j and the inputs outputted by those neurons in the *I*-layer via activation functions g_j (see [23]).

In [16], Yu and Cao studied the following differential equations with delay:

$$\begin{cases} \dot{x}_1(t) = -\mu_1 x_1(t) + c_{11} f_{11}(y_1(t-\tau_3)) + c_{12} f_{12}(y_2(t-\tau_3)), \\ \dot{x}_2(t) = -\mu_2 x_2(t) + c_{21} f_{21}(y_1(t-\tau_4)) + c_{22} f_{22}(y_2(t-\tau_4)), \\ \dot{y}_1(t) = -\mu_3 y_1(t) + d_{11} g_{11}(x_1(t-\tau_1)) + d_{12} g_{12}(x_2(t-\tau_2)), \\ \dot{y}_2(t) = -\mu_4 y_2(t) + d_{21} g_{21}(x_1(t-\tau_1)) + d_{22} g_{22}(x_2(t-\tau_2)) \end{cases} \tag{1.2}$$

☆This work is supported by National Natural Science Foundation of China (Nos. 10771215 and 10771094).

* Corresponding author. Tel./fax: +86 731 52680745.
E-mail addresses: xcj403@126.com (C. Xu), tangxh@mail.csu.edu.cn (X. Tang).

0925-2312/$ - see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.neucom.2010.09.002

and obtain the condition of the existence of Hopf bifurcation, a formula for determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions.

Motivated by the paper [16] and considering that when the number of neurons is large, the simplified model can reflect the really large neural networks more closely, we assume that there are three neurons in the I-layer and J-layer, respectively. Then we obtain the following six dimensional delayed bidirectional associative memory neural network:

$$\begin{cases} \dot{x}_1(t)=-\mu_1x_1(t)+c_{11}f_{11}(y_1(t-\tau_4))+c_{12}f_{12}(y_2(t-\tau_4))+c_{13}f_{13}(y_3(t-\tau_4)),\\ \dot{x}_2(t)=-\mu_2x_2(t)+c_{21}f_{21}(y_1(t-\tau_5))+c_{22}f_{22}(y_2(t-\tau_5))+c_{23}f_{23}(y_3(t-\tau_5)),\\ \dot{x}_3(t)=-\mu_3x_3(t)+c_{31}f_{31}(y_1(t-\tau_6))+c_{32}f_{32}(y_2(t-\tau_6))+c_{33}f_{33}(y_3(t-\tau_6)),\\ \dot{y}_1(t)=-\mu_4y_1(t)+c_{41}f_{41}(x_1(t-\tau_1))+c_{42}f_{42}(x_2(t-\tau_2))+c_{43}f_{43}(x_3(t-\tau_3)),\\ \dot{y}_2(t)=-\mu_5y_2(t)+c_{51}f_{51}(x_1(t-\tau_1))+c_{52}f_{52}(x_2(t-\tau_2))+c_{53}f_{53}(x_3(t-\tau_3)),\\ \dot{y}_3(t)=-\mu_6y_3(t)+c_{61}f_{61}(x_1(t-\tau_1))+c_{62}f_{62}(x_2(t-\tau_2))+c_{63}f_{63}(x_3(t-\tau_3)). \end{cases} \tag{1.3}$$

In this paper, we consider the model (1.3). In order to establish the main results for model (1.3), it is necessary to make the following assumptions:

$(H1)\quad f_{ij}\in C^k, f_{ij}(0)=0\quad (i=1,2,3,4,5,6;\ j=1,2,3;\ k=1,2,3,\ldots).$

$(H2)\quad \tau_1+\tau_4=\tau_2+\tau_5=\tau_3+\tau_6=\tau.$

The purpose of this paper is to discuss the stability and the properties of Hopf bifurcation of model (1.3). We would like to mention that there are few papers related to the high dimensional neural networks system with multiple delays. To the best of our knowledge, it is the first time to deal with the dynamical properties of six dimensional neural networks, especially the properties of Hopf bifurcation.

This paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of Hopf bifurcation at the equilibrium are studied. In Section 3, the direction of Hopf bifurcation and the stability and periodic of bifurcating periodic solutions on the center manifold are determined. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

2. Stability of the equilibrium and local Hopf bifurcations

Let

$$\begin{cases} u_1(t)=x_1(t-\tau_1),\\ u_2(t)=x_2(t-\tau_2),\\ u_3(t)=x_3(t-\tau_3),\\ u_4(t)=y_1(t),\\ u_5(t)=y_2(t),\\ u_6(t)=y_3(t), \end{cases} \tag{2.1}$$

then system (1.3) takes the following equivalent form

$$\begin{cases} \dot{u}_1(t)=-\mu_1u_1(t)+c_{11}f_{11}(u_4(t-\tau))+c_{12}f_{12}(u_5(t-\tau))+c_{13}f_{13}(u_6(t-\tau)),\\ \dot{u}_2(t)=-\mu_2u_2(t)+c_{21}f_{21}(u_4(t-\tau))+c_{22}f_{22}(u_5(t-\tau))+c_{23}f_{23}(u_6(t-\tau)),\\ \dot{u}_3(t)=-\mu_3u_3(t)+c_{31}f_{31}(u_4(t-\tau))+c_{32}f_{32}(u_5(t-\tau))+c_{33}f_{33}(u_6(t-\tau)),\\ \dot{u}_4(t)=-\mu_4u_4(t)+c_{41}f_{41}(u_1(t))+c_{42}f_{42}(u_2(t))+c_{43}f_{43}(u_3(t)),\\ \dot{u}_5(t)=-\mu_5u_5(t)+c_{51}f_{51}(u_1(t))+c_{52}f_{52}(u_2(t))+c_{53}f_{53}(u_3(t)),\\ \dot{u}_6(t)=-\mu_6u_6(t)+c_{61}f_{61}(u_1(t))+c_{62}f_{62}(u_2(t))+c_{63}f_{63}(u_3(t)). \end{cases} \tag{2.2}$$

By the hypothesis (H1), it is easy to see that (2.2) has a unique equilibrium $u_*(0,0,0,0,0,0)$. Under the hypotheses (H1) and (H2), the linear equation of (2.2) at $u_*(0,0,0,0,0,0)$ takes the form

$$\begin{cases} \dot{u}_1(t)=-\mu_1u_1(t)+\alpha_{11}u_4(t-\tau)+\alpha_{12}u_5(t-\tau)+\alpha_{13}u_6(t-\tau),\\ \dot{u}_2(t)=-\mu_2u_2(t)+\alpha_{21}u_4(t-\tau)+\alpha_{22}u_5(t-\tau)+\alpha_{23}u_6(t-\tau),\\ \dot{u}_3(t)=-\mu_3u_3(t)+\alpha_{31}u_4(t-\tau)+\alpha_{32}u_5(t-\tau)+\alpha_{33}u_6(t-\tau),\\ \dot{u}_4(t)=-\mu_4u_4(t)+\alpha_{41}u_1(t)+\alpha_{42}u_2(t)+\alpha_{43}u_3(t),\\ \dot{u}_5(t)=-\mu_5u_5(t)+\alpha_{51}u_1(t)+\alpha_{52}u_2(t)+\alpha_{53}u_3(t),\\ \dot{u}_6(t)=-\mu_6u_6(t)+\alpha_{61}u_1(t)+\alpha_{62}u_2(t)+\alpha_{63}u_3(t), \end{cases} \tag{2.3}$$

where $\alpha_{ij}=c_{ij}f'_{ij}(0)$ $(i=1,2,3,4,5,6,\ j=1,2,3)$. Then the associated characteristic equation of (2.3) is

$$\det\begin{pmatrix} \lambda+\mu_1 & 0 & 0 & -\alpha_{11}e^{-\lambda\tau} & -\alpha_{12}e^{-\lambda\tau} & -\alpha_{13}e^{-\lambda\tau}\\ 0 & \lambda+\mu_2 & 0 & -\alpha_{21}e^{-\lambda\tau} & -\alpha_{22}e^{-\lambda\tau} & -\alpha_{23}e^{-\lambda\tau}\\ 0 & 0 & \lambda+\mu_3 & -\alpha_{31}e^{-\lambda\tau} & -\alpha_{32}e^{-\lambda\tau} & -\alpha_{33}e^{-\lambda\tau}\\ -\alpha_{41} & -\alpha_{42} & -\alpha_{43} & \lambda+\mu_4 & 0 & 0\\ -\alpha_{51} & -\alpha_{52} & -\alpha_{53} & 0 & \lambda+\mu_5 & 0\\ -\alpha_{61} & -\alpha_{62} & -\alpha_{63} & 0 & 0 & \lambda+\mu_6 \end{pmatrix}=0 \tag{2.4}$$

which leads to the following form:

$$p_1(\lambda)+p_2(\lambda)e^{-\lambda\tau}+p_3(\lambda)e^{-2\lambda\tau}+p_4(\lambda)e^{-3\lambda\tau}=0, \tag{2.5}$$

where $p_1(\lambda),p_2(\lambda),p_3(\lambda),p_4(\lambda)$ are defined by Appendix A.

Multiplying $e^{\lambda\tau}$ on both sides of (2.5), it is easy to obtain

$$p_1(\lambda)e^{\lambda\tau}+p_2(\lambda)+p_3(\lambda)e^{-\lambda\tau}+p_4(\lambda)e^{-2\lambda\tau}=0. \tag{2.6}$$

Let $\lambda=i\omega_0$, $\tau=\tau_0$, and substituting this into (2.6), for the sake of simplicity, denote ω_0 and τ_0 by ω,τ, respectively, then (2.6) becomes

$$(A_1+iB_1)(\cos\omega\tau+i\sin\omega\tau)+A_2+iB_2+(A_3+iB_3)\\ \times(\cos\omega\tau-i\sin\omega\tau)+A_4(\cos2\omega\tau-i\sin2\omega\tau)=0, \tag{2.7}$$

where

$$A_i=\mathrm{Re}\{p_i(i\omega)\},\quad B_i=\mathrm{Im}\{p_i(i\omega)\},\ (i=1,2,3,4). \tag{2.8}$$

Separating the real and imaginary parts, we have

$$(A_1+A_3)\cos\omega\tau+(B_3-B_1)\sin\omega\tau+A_2=-A_4\cos2\omega\tau, \tag{2.9}$$

$$(B_1+B_3)\cos\omega\tau+(A_1-A_3)\sin\omega\tau+B_2=A_4\sin2\omega\tau. \tag{2.10}$$

Squaring both sides of (2.9) and (2.10), and adding them up gives

$$[(A_1+A_3)\cos\omega\tau+(B_3-B_1)\sin\omega\tau+A_2]^2\\ +[(B_1+B_3)\cos\omega\tau+(A_1-A_3)\sin\omega\tau+B_2]^2=A_4^2. \tag{2.11}$$

According to $\sin\omega\tau=\pm\sqrt{1-\cos^2\omega\tau}$, we consider the two cases:

(I) If $\sin\omega\tau=\sqrt{1-\cos^2\omega\tau}$, then (2.11) takes the following form:

$$[(A_1+A_3)\cos\omega\tau+(B_3-B_1)\sqrt{1-\cos^2\omega\tau}+A_2]^2\\ +[(B_1+B_3)\cos\omega\tau+(A_1-A_3)\sqrt{1-\cos^2\omega\tau}+B_2]^2=A_4^2. \tag{2.12}$$

It is easy to see that (2.12) is equivalent to

$$q_1\cos^4\omega\tau+q_2\cos^3\omega\tau+q_3\cos^2\omega\tau+q_4\cos\omega\tau+q_5=0, \tag{2.13}$$

where

$$q_1=[2(A_1+A_3)(B_3-B_1)+2(B_1+B_3)(A_1-A_3)]^2+[4(A_1A_3+B_1B_3)]^2,$$

$$q_2=2\{[2(A_1+A_3)(B_3-B_1)+2(B_1+B_3)(A_1-A_3)][2A_2(B_3-B_1)\\ +2B_2(A_1-A_3)]+[2A_2(B_3-B_1)+2B_2(A_1-A_3)][4(A_1A_3+B_1B_3)]\},$$

$$q_3=[2A_2(A_1+A_3)+2B_2(B_1+B_3)]^2+[2A_2(B_3-B_1)+2B_2(A_1-A_3)]^2\\ -[2(A_1+A_3)(B_3-B_1)+2(B_1+B_3)(A_1-A_3)]^2\\ -2[A_4^2+B_4^2-A_2^2-B_2^2-(B_3-B_1)^2-(A_1-A_3)^2][4(A_1A_3+B_1B_3)],$$