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Global exponential stability in Lagrange sense for periodic neural networks with various activation functions

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ABSTRACT

In this paper, global exponential stability in Lagrange sense for periodic neural networks with various activation functions is further studied. By constructing appropriate Lyapunov-like functions, we provide easily verifiable criteria for the boundedness and global exponential attractivity of periodic neural networks. These theoretical analysis can narrow the search field of optimization computation, associative memories, chaos control and provide convenience for applications.

1. Introduction

Lyapunov stability is one of the important properties of dynamic systems. From a system-theoretic point of view, the global stability of neural networks is a very interesting issue for research because of the special nonlinear structure of neural networks. From a dynamical system point of view, globally stable neural networks in Lyapunov sense are monostable systems, which have a unique equilibrium attracting all trajectories asymptotically, more specific results are referred to [1–10,16–18]. In many other applications, however, monostable neural networks have been found to be computationally restrictive and multistable dynamics are essential to deal with the important neural computations desired. In these circumstances, neural networks are no longer globally stable and more appropriate notions of stability are need to deal with multistable systems. In this context, many researchers focus on the Lagrange stability. It is noted that unlike Lyapunov stability, Lagrange stability refers to the stability of the total system, not the stability of the equilibriums, because the Lagrange stability is considered on the basis of the boundedness of solutions and the existence of global attractive sets (see [11–15,19–21]). We also note that Lagrange stability has attracted phenomenal worldwide attention. In [11,12], Liao et al. apply Lyapunov functions to study Lagrange stability for recurrent neural networks. In [13], Yang and Cao consider stability in Lagrange sense of a class of feedback neural networks for optimization problems. In [14,15], Lagrange stability is discussed for Cohen–Grossberg neural networks. At present, although a series of results for periodic neural networks are obtained (see [1–10,16–18]), the stability analysis in the Lagrange sense for periodic neural networks does not appear. So from the theoretical and application views, it is necessary to study the stable properties in the Lagrange sense for periodic neural networks.

Moreover, in conducting stability analysis of a neural network, the conditions to be imposed on the neural network are determined by the characteristics of activation function as well as network parameters. As we know, when neural networks are designed for problem solving, it is desirable for their activation functions to be general. To facilitate the design of neural networks, it is important that the neural networks with general activation functions are studied. The generalization of activation functions will provide a wider scope for neural network designs and applications. Motivated by the above discussions, our objective in this paper is to study the global exponential stability (GES) in Lagrange sense for periodic neural networks with various activation functions, which include both bounded and unbounded activation functions. We provide verifiable criteria for the boundedness of the networks and the existence of globally exponentially attractive (GEA) sets by constructing appropriate Lyapunov-like functions. It is believed that the results are significant and useful for the design and applications of the periodic neural networks.

This paper is organized as follows. In Section 2, we define the notions of GEA sets and GES in Lagrange sense, and give two preliminary results that will be used in the proofs of the main results. In Section 3, we provide several sufficient conditions for the GES in Lagrange sense of periodic neural networks with bounded activation functions. GES in Lagrange sense of Lurie-type activation functions is studied in Section 4. In Section 5, a numerical example is given to

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illustrate the applications of the results. Finally, in Section 6 we give the conclusion.

2. Preliminaries

Consider the following neural network model of the form:

$$\dot{x}_i(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n}(a_{ij}(t)g_j(x_j(t)) + b_{ij}(t)g_j(x_j(t-\tau_j))) + I_i(t), \quad t \geq 0, \tag{1}$$

where $i = 1,2,\ldots,n$, $d_i(t) > 0$, $x_i(t)$ is the state variable of the ith neuron, $I_i(t) \in C(R,R)$ is an external input, $a_{ij}(t)$ and $b_{ij}(t)$ are connection weights from neuron i to neuron j, τ_j corresponds to the transmission delay and satisfies $0 \leq \tau_j \leq \tau$ (τ is a constant), $g_i(\cdot) \in C(R,R)$ is the neuron activation function, the initial conditions associated with (1) is of the form $x_i(s) = \psi_i(s)$ for $s \in [-\tau,0]$.

Throughout the paper, we assume that $d_i(t)$, $a_{ij}(t)$, $b_{ij}(t)$ and $I_i(t)$ are continuous $\omega-$periodic functions, i.e., $d_i(t+\omega) = d_i(t)$, $a_{ij}(t+\omega) = a_{ij}(t)$, $b_{ij}(t+\omega) = b_{ij}(t)$ and $I_i(t+\omega) = I_i(t)$.

In this paper, we will consider two classes of activation functions for the neural network model (1). To this end, we define the vector function $g \in C(R^n,R^n)$ by $g(x) := (g_1(x_1), g_2(x_2), \ldots, g_n(x_n))$, where $x = (x_1,x_2,\ldots,x_n) \in R^n$.

Firstly, we consider the bounded activation functions, which can be given in the form

$$\mathcal{B} \triangleq \{g(\cdot)|g_i \in C(R,R), \exists k_i > 0, |g_i(x_i)| \leq k_i, \forall x_i \in R, i = 1,2,\ldots,n\},$$

where the constants k_i, $i = 1,2,\ldots,n$, are generally called to be saturation constants.

Secondly, we consider the Lurie-type activation functions, which can be given in the form

$$\mathcal{F} \triangleq \{g(\cdot)|g_i \in \mathcal{K}, \exists k_i > 0, x_i g_i(x_i) \leq k_i x_i^2, \forall x_i \in R, i = 1,2,\ldots,n\},$$

where

$$\mathcal{K} \triangleq \{\phi \in C(R,R)|s\phi(s) \geq 0 \text{ and } D^+\phi(s) \geq 0, s \in R\},$$

and the constants k_i, $i = 1,2,\ldots,n$, are generally called to be Lurie constants.

For convenience, we introduce the following notations:

Let C be the Banach space of continuous functions $\psi : [-\tau,0] \to R^n$ with the norm $\|\psi\| = \sup_{s \in [-\tau,0]}|\psi(s)|$. For a given constant $H > 0$, C_H is defined as the subset $\{\psi \in C : \|\psi\| \leq H\}$. Let $\mathcal{CCF}^+$ be the set of all non-negative continuous functionals $K : C \to [0,+\infty)$, mapping bounded sets in C into bounded sets in $[0,+\infty)$. For any initial condition $\psi \in C$, the solution of (1) that starts from the initial condition ψ will be denoted by $x(t;\psi)$. If there is no need to emphasize the initial condition, any solution of (1) will also simply be denoted by $x(t)$. For any continuous bounded function $h(t)$, $t \geq 0$, we write $|\overline{h}| = \sup_{t \geq 0}|h(t)|$ and $\underline{h} = \inf_{t \geq 0} h(t)$.

Definition 2.1 (*Wang et al.* [14]). Network (1) is said to be uniformly stable in Lagrange sense (or uniformly bounded), if for any $H > 0$, there exists a constant $K = K(H) > 0$ such that $|x(t;\psi)| < K$ for all $\psi \in C_H$ and $t \geq 0$.

Definition 2.2 (*Liao et al.* [11]). Let $\Omega \subset R^n$ be a compact set in R^n and the complement of Ω by $R^n \backslash \Omega$. For any $x \in R$, $\rho(x,\Omega) := \inf_{y \in \Omega}|x-y|$ is the distance between x and Ω, a compact set $\Omega \subset R^n$ is said to be a global attractive set of network (1), if for every solution $x(t)$ such that $x(t) \in R^n \backslash \Omega, t \geq 0$, we have $\lim_{t \to +\infty}\rho(x(t),\Omega) = 0$.

Obviously, if network (1) has a global attractive set, it is ultimately bounded.

In many neural network applications, the rate of convergence or attractivity is very important in improving computational performance. In the following, we introduce the notion of globally exponentially attractive (GEA) sets.

Definition 2.3 (*Wang et al.* [14]). A compact set $\Omega \subset R^n$ is said to be a GEA set of (1) (in strong sense), if there exist a constant $\Lambda > 0$ and a continuous functional $K \in \mathcal{CCF}^+$ such that for every solution $x(t)$ with $x(t) \in R^n \backslash \Omega, t \geq 0$, we have $\rho(x(t),\Omega) \leq K(\psi)\exp\{-\Lambda t\}$ for all $t \geq 0$.

It is easy to see that the above definition for a GEA set is often inconvenient to use, and the main difficulty lies in the use of the distance function $\rho(x,\Omega)$, which seems hard to be linked to the exponential decay function. In the following, we give two definitions for a GEA set in a weaker form. And these definitions explicitly involve in positive definite functions, hence they are typically suitable in employing Lyapunov-like functions to analyze global exponential attractivity of neural network models.

Definition 2.4 (*Liao et al.* [11]). If there exist a radially unbounded and positive definite function $V(x)$, a continuous functional $K \in \mathcal{CCF}^+$, positive constants ℓ and α, such that for any solution $x(t) = x(t;\psi)$ of (1), $V(x(t)) > \ell, t \geq 0$, implies

$$V(x(t)) - \ell \leq K(\psi)\exp\{-\alpha t\},$$

then network (1) is said to be globally exponentially attractive with respect to V, and the compact set $\Omega := \{x \in R^n | V(x) \leq \ell\}$ is called to be a GEA set of (1).

Definition 2.5 (*Liao et al.* [12]). If there exist radially unbounded and positive definite functions $V_i(x_i), x_i \in R$, continuous functionals $K_i \in \mathcal{CCF}^+$, positive constants ℓ_i and α_i, such that for any solution $x(t) = x(t;\psi)$ of (1), $V_i(x_i(t)) > \ell_i, t \geq 0$, implies

$$V_i(x_i(t)) - \ell_i \leq K_i(\psi)\exp\{-\alpha_i t\}, \quad i = 1,2,\ldots,n,$$

then network (1) is said to be globally exponentially attractive with respect to $(V_1,V_2,\ldots,V_n)$, and the compact set $\Omega := \{x \in R^n | V_i(x_i) \leq \ell_i\}$ is called to be a GEA set of (1).

Definition 2.6 (*Wang et al.* [14]). Network (1) is called globally exponentially stable (GES) in Lagrange sense, if it is both uniformly stable in Lagrange sense and globally exponentially attractive. If there is a need to emphasize the Lyapunov-like functions, the network will be called globally exponentially stable in Lagrange sense with respect to V or $(V_1,V_2,\ldots,V_n)$.

We end this section with two preliminary results, which will be used in the proofs of the main results.

Lemma 2.1 (*Wang et al.* [14]). *Let $G \in C([t_0,\infty),R)$, and there exist positive constants α and β such that*

$$D^+G(t) \leq -\alpha G(t) + \beta, \quad t \geq t_0,$$

then

$$G(t) - \frac{\beta}{\alpha} \leq \left(G(t_0) - \frac{\beta}{\alpha}\right)\exp\{-\alpha(t-t_0)\}, \quad t \geq t_0.$$

In particular, if $G(t) \geq \beta/\alpha$, $t \geq t_0$, then $G(t)$ exponentially approaches β/α as t increases.

The following lemma provides a key step in proving global exponential attractivity for system (1).

Lemma 2.2. *Let $x(t) := x(t;\psi)$ be a solution of (1) and g_i be given in (1) with $x_i g_i(x_i) \geq 0$, $x_i \in R$, $i = 1,2,\ldots,n$. Define the function*

$$V(t) = \sum_{i=1}^{n}\int_0^{x_i(t)} g_i(y)\,\mathrm{d}y + \sum_{i=1}^{n}\int_{t-\tau_i}^{t} g_i^2(x_i(s))\,\mathrm{d}s, \quad t \geq 0,$$