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Exponential stability of numerical solutions to stochastic delay Hopfield neural networks

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ABSTRACT

The main aim of this paper is to investigate the exponential stability of the Euler method and the semi-implicit Euler method for stochastic delay Hopfield neural networks. The definition of MS-stability and GMS-stability of these two numerical methods is introduced. Under the conditions which guarantee the stability of the analytical solution, the Euler scheme is proved to be MS-stable and the semi-implicit scheme is to be MS-stable and GMS-stable. An example is given for illustration.

1. Introduction

Stochastic delay Hopfield neural networks have been widely used to model many of the phenomena arising in such areas as associative content-addressable memories, pattern recognition and optimization. One of the important and interesting problems in the analysis of stochastic delay Hopfield neural networks is their exponential stability. The exponential stability of stochastic delay Hopfield neural networks has been studied by many authors and we here mention Blythe et al. [1], Zhou and Wan [2], Wan and Sun [3], Sun and Cao [4], Wang et al. [5] and references therein.

Most of stochastic delay Hopfield neural networks, similar to stochastic delay differential equations, do not have explicit solutions. Thus appropriate numerical approximation schemes such as the Euler scheme are needed to apply stochastic delay Hopfield neural networks in practice or to study their properties. To the best of our knowledge, there has been little work on the exponential stability of numerical methods for stochastic delay Hopfield neural networks, although there are many papers concerned with the numerical solutions to stochastic delay differential equations [6–10].

Motivated by the importance to study the numerical problem of stochastic delay Hopfield neural networks and the issue of their global stability for strong solutions to stochastic delay Hopfield neural networks, in this paper, we consider the stability of numerical solutions to stochastic delay Hopfield neural networks of the form

$$\begin{cases} dx_i(t) = \left[-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_j)) \right] dt + \sigma_i(x_i(t))\, dW_i(t), \\ x_i(t) = \xi_i(t), \quad -\tau_i \le t \le 0, \end{cases} \tag{1}$$

where $i = 1, 2, \ldots, n$, $t \ge 0$. In the above model, $n \ge 1$ is the number of neurons in the network, x_i is the state variable of the i th neuron at time t, f_j and g_j denote the output of the j th unit at time t and $t-\tau_j$, respectively, σ_i are continuous functions, c_i represents the rate with which the i th unit will reset its potential to the resting state in isolation when discounted from the network and the external stochastic perturbation, and is a positive constant; a_{ij} and b_{ij} weight

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the strength of the j th on the i th unit, τ_j is the transmission delay which is a nonnegative constant.

In this paper, both the Euler method and the semi-implicit Euler method are considered. The main aim of this paper is to show that the Euler method applied to Eq. (1) is MS-stable, and the semi-implicit Euler method applied to Eq. (1) is MS-stable and GMS-stable under the conditions which guarantee the stability of the analytical solution. Our work differs from Refs. [11–14] in that: (1) two numerical schemes are considered; (2) τ_j may be different.

The paper is organized as follows. In Section 2, we shall introduce some notations and hypotheses of Eq. (1), and give some properties of its analytical solution. In Section 3, we shall prove the MS-stability of the Euler numerical solution to Eq. (1). In Section 4, we shall prove the MS-stability and GMS-stability of the semi-implicit Euler numerical solution to the special form of Eq. (1). An example is provided to illustrate our theory in Section 5. Conclusion is given in Section 6.

2. Analysis of analytical solution

Throughout this paper, we let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). $x(t) = (x_1(t), \ldots, x_n(t))^T$. Let $C([-\tau, 0]; R^n)$ be the family of continuous function ϕ from $[-\tau, 0]$ to R^n with the norm $\|\phi\| = \sup_{-\tau\leq t\leq 0}|\phi(t)|$, while $|\cdot|$ is the Euclidean norm in R^n. Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ the family of all bounded $\mathcal{F}_0-$measurable $C([-\tau, 0]; R^n)-$valued random variables. We assume $\xi = \{\xi(t) = (\xi_1(t), \ldots, \xi_n(t))^T\} \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$.

In order to obtain the stability of Eq. (1), We impose the following standing hypotheses:

(H1) $f_j(0) = g_j(0) = \sigma_i(0) = 0$. f_j, g_j and σ_i satisfy the global Lipschitz condition with Lipschitz constants $\alpha_j > 0$, β_j and $L_i > 0$, respectively.

Under the above assumptions, Eq. (1) has a unique strong solution $x(t) = x(t;\xi)$ on $t \geq 0$ and $x(t)$ is a measurable, sample-continuous and $\mathcal{F}_t-$adapted process. This result can be found in [15]. Clearly, Eq. (1) admits the trivial solution $x \equiv 0$.

For the purpose of establishing the stability condition of Eq. (1), the following result is required.

Lemma 1. *If Eq.* (1) *satisfies the conditions* (H1) *and*

(H2) *For* $i = 1, 2, \ldots, n$,

$$-2c_i + \sum_{j=1}^n |a_{ij}|\alpha_j + \sum_{j=1}^n |b_{ij}|\beta_j + \sum_{j=1}^n |a_{ji}|\alpha_i + \sum_{j=1}^n |b_{ji}|\beta_i + L_i^2 < 0. \qquad (2)$$

Then Eq. (1) *is exponentially stable in mean square. That is, there exists a pair of positive constants λ and M_0 such that for any ξ*

$$E|x(t;\xi)|^2 \leq M_0 e^{-\lambda t} E|\xi|^2, \quad t \geq 0.$$

The proof of this lemma is found in [2].

3. Stability of Euler numerical solution

For Eq. (1), the discrete Euler approximate solution is defined by

$$y_i^{k+1} = y_i^k + \left[-c_i y_i^k + \sum_{j=1}^n a_{ij} f_j(y_j^k) + \sum_{j=1}^n b_{ij} g_j(y_j^{k-m_j})\right] h + \sigma_i(y_i^k)\Delta W_i^k, \qquad (3)$$

where $i = 1, 2, \ldots, n$, h $(0 < h < 1)$ is a stepsize which satisfies $\tau_j = m_j h$ for a positive integer m_j, and $t_k = kh$, y_i^k is an approximation to $x_i(t_k)$, if $t_k \leq 0$, we have $y_i^k = \xi_i(t_k)$. Moreover, the increments $\Delta W_i^k = W_i(t_{k+1}) - W_i(t_k)$ are normal distribution with mean zero and variance h.

Suppose that the following condition is satisfied.

(H3)

$$\sum_{j=1}^n |a_{ij}|\alpha_j + \sum_{j=1}^n |b_{ij}|\beta_j \leq \sum_{j=1}^n |a_{ji}|\alpha_i + \sum_{j=1}^n |b_{ji}|\beta_i.$$

Definition 2. Under the conditions (H1)–(H3), a numerical method is said to be mean square stable (MS-stable), if there exists a $h_0 > 0$, such that any application of the method to Eq. (1) generates numerical approximations y_i^k, which satisfy

$$\lim_{k\to\infty} E|y_i^k|^2 = 0, \quad i = 1, 2, \ldots, n$$

for all $h \in (0, h_0)$ with $h = \tau_j/m_j$.

Now we are in position to give one of the main results of this paper.

Theorem 3. *Under the conditions* (H1)–(H3), *the Euler method applied to Eq.* (1) *is MS-stable.*

Proof. From Eq. (3) we have

$$y_i^{k+1} = [(1-c_i h)y_i^k + \sigma_i(y_i^k)\Delta W_i^k] + h\sum_{j=1}^n a_{ij} f_j(y_j^k) + h\sum_{j=1}^n b_{ij} g_j(y_j^{k-m_j}). \qquad (4)$$

Squaring both sides of Eq. (4), we have

$$\begin{aligned}(y_i^{k+1})^2 = {} & [(1-c_i h)y_i^k + \sigma_i(y_i^k)\Delta W_i^k]^2 + h^2\left[\sum_{j=1}^n a_{ij} f_j(y_j^k)\right]^2 \\ & + h^2\left[\sum_{j=1}^n b_{ij} g_j(y_j^{k-m_j})\right]^2 + 2h[(1-c_i h)y_i^k \\ & + \sigma_i(y_i^k)\Delta W_i^k]\left[\sum_{j=1}^n a_{ij} f_j(y_j^k)\right] + 2h[(1-c_i h)y_i^k \\ & + \sigma_i(y_i^k)\Delta W_i^k]\left[\sum_{j=1}^n b_{ij} g_j(y_j^{k-m_j})\right] \\ & + 2h^2\left[\sum_{j=1}^n a_{ij} f_j(y_j^k)\right]\left[\sum_{j=1}^n b_{ij} g_j(y_j^{k-m_j})\right]. \end{aligned} \qquad (5)$$

It follows from the inequality $2abxy \leq |ab|(x^2+y^2)$ that

$$\begin{aligned}(y_i^{k+1})^2 \leq {} & (1-c_i h)^2 (y_i^k)^2 + \sigma_i^2(y_i^k)(\Delta W_i^k)^2 + 2(1-c_i h)(y_i^k)\sigma_i(y_i^k)\Delta W_i^k \\ & + h^2\sum_{j=1}^n |a_{ij}|\alpha_j\left(\sum_{r=1}^n |a_{ir}|\alpha_r\right)(y_j^k)^2 \\ & + h^2\sum_{j=1}^n |b_{ij}|\beta_j\left(\sum_{r=1}^n |b_{ir}|\beta_r\right)(y_j^{k-m_j})^2 \\ & + h\sum_{j=1}^n |(1-c_i h)a_{ij}|\alpha_j[(y_i^k)^2 + (y_j^k)^2] \\ & + 2h\sigma_i(y_i^k)\Delta W_i^k\left[\sum_{j=1}^n a_{ij} f_j(y_j^k)\right] \\ & + h\sum_{j=1}^n |(1-c_i h)b_{ij}|\beta_j[(y_i^k)^2 + (y_j^{k-m_j})^2] \\ & + 2h\sigma_i(y_i^k)\Delta W_i^k\left[\sum_{j=1}^n b_{ij} g_j(y_j^{k-m_j})\right] \end{aligned}$$