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Efficient and accurate computation of model predictive control using pseudospectral discretization

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ABSTRACT

The model predictive control (MPC) is implemented by repeatedly solving an open loop optimal control problem (OCP). For the real-time implementation, the OCP is often discretized with evenly spaced time grids. This evenly spaced discretization, however, is accurate only if sufficiently small sampling time is used, which leads to heavy computational load. This paper presents a method to efficiently and accurately compute the continuous-time MPC problem based on the pseudospectral discretization, which utilizes unevenly spaced collocation points. The predictive horizon is virtually doubled by augmenting a mirrored horizon such that denser collocation points can be used towards the current time step, and sparser points can be used towards the end time of predictive horizon. Then, both state and control variables are approximated by Lagrange polynomials at only a half of LGL (Legendre–Gauss–Lobatto) collocation points. This implies that high accuracy can be achieved with a much less number of collocation points, which results in much reduced computational load. Examples are used to demonstrate its advantages over the evenly spaced discretization.

1. Introduction

Model predictive control (MPC) is implemented by repeatedly solving an open-loop optimal control problem (OCP) and using the first element of the optimized control sequence as the current control action [1]. This class of controllers has received wide applications in the process industries since 1980s [2]. The main advantage of MPC is its ability to explicitly handle constraints on controls and states and achieve optimized control inside admissible sets. Generally, the plant dynamics are described in the continuous time representation by resorting to first principle equations. For the real-life implementation, however, discrete-time descriptions are required, and the continuous time problem must be converted into a discrete time problem for the computer environment [3].

The typical approach for this conversion is to discretize the plant model, constraints, and cost function at evenly spaced time points. The evenly spaced discretization approach has been widely used in the literature, e.g. Chen and Allgower [4], Muske and Badgwell [5], Qin and Badgwell [6] and Scattolini [7]. In these studies, the system behaviors between two consecutive sampling points are usually approximated by using either zero order hold (ZOH) or first order hold (FOH), which are the two lowest order approximation methods [8,9]. For reduced computational efforts, some studies used discretization methods with unevenly spaced points, e.g. moving blocking strategy [10,11]. They, however, share the same approximation methods, ZOH and FOH, which leads to significant state prediction and control errors. Hence, small sampling time is necessary in order to achieve sufficiently high levels of accuracy, which results in high computational load.

In this study, we will discretize the continuous-time MPC with unevenly spaced grid points. The unevenly spaced discretization relies on the known pseudospectral approach, which enables to use high order polynomials to approximate states and controls. The high order approximation provides more flexibility to describe the system behavior between two points, thus effectively avoiding the shortcomings of lower order counterpart. In theory, the discretized NLP converges to the OCP at a spectral rate with the number of collocation points [12].

The pseudospectral method was first applied to optimal control problems in the late 1980s. The use of Chebyshev polynomial as interpolation basis is the first method [13]. Recently, the majority have employed Lagrange polynomials as basis functions, which are often categorized into three types: Legendre–Gauss–Lobatto (LGL) method, Legendre–Gauss (LG) method, and Legendre–Gauss–Radau (LGR) method [12]. The first type uses the family of Lobatto quadrature. The collocation points of LGL, defined on the closed interval $[-1,1]$, are the roots of the

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derivative of N th-degree Legendre polynomial, i.e., $\dot{L}_N(t)$, together with -1 and 1. Several variants has been presented before, including [14,15] and [16]. A more general version is the Jacobi pseudospectral method presented by [17]. The second type uses the family of Gauss quadrature [18–20]. The collocation points of LG, defined on the open interval $(-1, 1)$, are the zeros of N th-degree Legendre polynomial $L_N(t)$, so that the endpoints of integration interval are not included. The third type uses the family of Radau quadrature [21,22]. The collocation points of LGR, defined on the half-open interval $(-\mathbf{1}, \mathbf{1}]$, are the roots of $L_{N-1}(t)+L_N(t)$, which only contains one end point. Upon cursory examination it might appear as if LGL, LG and LGR collocation are essentially similar, with only minor difference on whether contains end points. It has been shown by Garg et al. that the differences between three schemes are not merely cosmetic [31]. The LGL collocation leads to a different mathematical form as compared with either LG or LGR. As a result, LGL has different convergence properties from LG and LGR. As pointed out by Garg et al., the main differences are: (1) The differentiation matrices of LG and LGR are rectangular and full-rank, whereas that of LGL is square and singular. Therefore, LG and LGR can be written equivalently in either differential or implicit integral form while LGL does not have an equivalent implicit integral from. (2) The differentiation matrix of LGL is rank deficient. This rank-deficiency leads to a transformed adjoint system that is also rank-deficient, which can yield dual solutions that oscillate about the true solution. Conversely, LG and LGR lead to full-rank transformed adjoint systems which in turn yield approximations that converge to the true solution. (3) There is a fundamental difference on costate estimation between LGL and LGR/LG. The aforementioned oscillatory property also affects the estimation of costate estimation, which naturally attributes to the null space of the LGL transformed adjoint system. This is because that the discrete costate dynamics form a linear system of equations in terms of Lagrange multiplier. In LGL, the matrix in the equations has a null space and therefore there exists a infinite number of solutions to LGL costate dynamics. Despite the null space in LGL, many numerical examples have demonstrated that LGL has convergent approximations to state and control. The convector mapping theorem by Gong et al. [24] has pointed out that any solution of first-order optimality condition for continuous system approximately satisfies the first-order optimality condition for discrete LGL problem, and therefore the error tends to zero as $N \to \infty$. Moreover, Gong et al. [24] proposes a closure condition for selecting a good approximation to continuous costate from the infinite number of solutions.

The main study of this paper is to discretize the continuous-time MPC problem by converting the open loop OCP into NLP using the pseudospectral discretization. This unevenly spaced discretization enables more efficient and accurate implementations of MPC over evenly spaced counterpart. This paper mainly relies on the LGL collocation scheme, which makes possible to directly apply initial solution as control input and add constraints for terminal points in the predictive horizon [26]. The remainder of this paper is organized as follows: Section 2 reviews the continuous-time MPC problem. Then, its predictive horizon is designed to be augmented by a mirrored horizon in Section 3. The Legendre pseudospectral method is applied to discretize of the augmented OCP in Section 4. Section 5 renews the known covector mapping principle for the augmented problem; Illustrative examples are given in Section 6. Section 7 concludes this paper.

2. Continuous-time MPC problem

Let us consider a class of nonlinear continuous-time systems described by

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u} \in \mathbb{R}^m$ is the control, $f:(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^n$ is the mapping function. Note that there exists a dual pair $(\boldsymbol{x}, \boldsymbol{u}) \in (\mathbb{X}, \mathbb{U}) \in (\mathbb{R}^n, \mathbb{R}^m)$ which satisfies $f(\boldsymbol{x}, \boldsymbol{u}) = 0$. The sets $\mathbb{X} \in \mathbb{R}^n$ and $\mathbb{U} \in \mathbb{R}^m$ are admissible box constraints for the state and input. For narrative convenience, the predictive horizon is assumed to range from $t=-1$ to $t=0$ for the sake of simplicity, where $t=-1$ is the current time. The optimal control problem for the predictive horizon (**Problem A**) is as follows.

$$\min_{\boldsymbol{u}(t), t \in [-1,0]} J = \varphi(\boldsymbol{x}(0)) + \int_{-1}^{0} L(\boldsymbol{x}(t), \boldsymbol{u}(t)) \mathrm{d}t \tag{2}$$

Subject to

$$\begin{aligned} &\boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \le 0, \\ &\boldsymbol{x}(-1) = \boldsymbol{x}_0, \\ &\boldsymbol{\varphi}(\boldsymbol{x}(0)) \le 0, \\ &\boldsymbol{x}(t) \in \mathbb{X} \in \mathbb{R}^n, \\ &\boldsymbol{u}(t) \in \mathbb{U} \in \mathbb{R}^m. \end{aligned} \tag{3}$$

In **Problem A**, $t=-1$ is the initial time and $t=0$ is the final time, $L:(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ and $\phi: \mathbb{R}^n \to \mathbb{R}$ are non-negative functions of $(\boldsymbol{x}, \boldsymbol{u})$ and $\boldsymbol{x}(0)$, $\boldsymbol{h}:(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^q$ and $\boldsymbol{\varphi}:(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^l$ reflects the constraint on the terminal state. The length of prediction horizon is 1, over which the cost function is optimized. The problem is solved numerically to obtain the optimal control trajectory $\boldsymbol{u}^*(t), t \in [-1, 0]$, and the value of $\boldsymbol{u}^*(t)$ at $t=-1$ is used for the current time step control.

3. Doubled horizon: augmented by mirrored horizon

In the model predictive control defined on the interval $[-1, 1]$ as shown in Fig. 1, only the first point ($t=-1$) of optimized sequence in the predictive horizon $[-1, 0]$ is implemented as control input. Thus the accuracy around -1 (beginning of predictive horizon) is more important than that around $t=0$ (end of predictive horizon). Meanwhile, a point deployment such as “dense beginning and sparse end” is beneficial to enhance the accuracy of MPC. Similar technique has been used in many MPC applications, in which “move-blocking” is called [32,33]. The move-blocking scheme has two mechanisms to reduce computing complexity, of which one is to reduce the degrees of freedom by fixing some controls to be constant, and the other is to use thicker points in the beginning of predictive horizon and thinner points at the end. In contrast, the standard pseudospectral framework has the same point density at $t=-1$ and $t=+1$, which is a less efficient deployment of collocation points for MPC. In order to more effectively use LGL, the predictive horizon is virtually augmented by a mirrored horizon, which allows reducing the collocation points into half. The length of the mirrored horizon is designed to be identical to that of the predictive horizon. The states and inputs in mirrored horizon are then assumed to be symmetric to those of the predictive horizon, as illustrated in Fig. 1. The optimal control problem over the entire (predictive and mirrored) horizon is called an augmented problem, defined as **Problem B**. The mirrored strategy is designed to use denser points around initial time (i.e., $t=-1$) and sparser points around terminal time (i.e., $t=0$), which can ensure better accuracy around -1 with half-LGL collocation as compared with full LGL collocation under identical number of points. The mirrored strategy can generate similar efficiency-enhancing effect to the move-blocking technique.