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The Laplacian polynomial and Kirchhoff index of graphs based on R-graphs

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ABSTRACT

Let $R(G)$ be the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the corresponding edge. Let $I(G)$ be the set of newly added vertices. The R-vertex corona of G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the ith vertex of $V(G_1)$ to every vertex in the ith copy of G_2. The R-edge corona of G_1 and G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the ith vertex of $I(G_1)$ to every vertex in the ith copy of G_2. Liu et al. gave formulae for the Laplacian polynomial and Kirchhoff index of $RT(G)$ in [19]. In this paper, we give the Laplacian polynomials of $G_1 \odot G_2$ and $G_1 \ominus G_2$ for a regular graph G_1 and an arbitrary graph G_2; on the other hand, we derive formulae and lower bounds of Kirchhoff index of these graphs and generalize the existing results.

1. Introduction

Throughout this paper, all graphs considered are simple and undirected. Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)=\{v_1,v_2,\dots,v_n\}$ and edge set $E(G)=\{e_1,e_2,\dots,e_m\}$. The adjacency matrix of G, denoted by $A(G)$, is an $n\times n$ symmetric matrix such that $a_{ij}=1$ if vertices v_i and v_j are adjacent and 0 otherwise. Let $d_i=d_G(v_i)$ be the degree of vertex v_i in G and $D(G)=diag(d_1,d_2,\dots,d_n)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is defined as $L(G)=D(G)-A(G)$. Denoted by $P_G(x)$ and $\mu_G(x)$ the adjacent characteristic polynomial $\det(xI-A(G))$ and the Laplacian characteristic polynomial $\det(xI-L(G))$ of G, respectively. Since $A(G)$ and $L(G)$ are all real symmetric matrices, their eigenvalues are real numbers. So we can assume that $\lambda_1(G)\ge\lambda_2(G)\ge\cdots\ge\lambda_n(G)$ (resp., $0=\mu_1(G)\le\mu_2(G)\le\cdots\le\mu_n(G)$)are the adjacency (resp., Laplacian) eigenvalues of G. The collection of the adjacency (resp., Laplacian) eigenvalues of G together with their multiplicities forms the adjacency (resp., Laplacian) spectrum of G. For other undefined notations and terminology from graph theory, the readers may refer to [1] and the references therein. There are many applications for Laplacian eigenvalues of graphs. For example, there are many problems in physics and chemistry where the Laplacian eigenvalues play the central role. The Laplacian eigenvalues are in the segmentation of the combination optimization, method of design, parallel algorithm, solving linear systems, clustering and other aspects of a wide range of applications. See [32,33].

In 1993, Klein and Randić [2] introduced a distance function named resistance distance on the basis of electrical network theory. They view a graph as an electrical network each edge of the graph is assumed to be a unit resistor, then take the resistance distance between vertices to be the effective resistance between them. Let G be a simple graph with the vertex set $V(G)=\{v_1,v_2,\dots,v_n\}$, and r_{ij} denote the effective resistance distance between vertices v_i and v_j as computed with Ohm's law when all the edges of G are considered to be unit resistors. The sum of resistance distance $Kf(G)=\sum_{i<j}r_{ij}(G)$ was proposed in [1], later called the Kirchhoff index of G in [3]. In electric theory, it is of interest to compute the effective resistance between any pair of vertices of a network, as well as the Kirchhoff index. The Kirchhoff index was introduced in chemistry as a better alternative to other parameters used for discriminating different molecules with similar shapes and structures. See [2]. The resistance distance and the Kirchhoff index attracted extensive attention due to its wide applications in physics, chemistry, etc. See [4–9]. For more

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information on resistance distance and Kirchhoff index of graphs, the readers are referred to the papers [7–9].

In [10], new graph operations based on $R(G)$ graphs: R-vertex corona and R-edge corona, are introduced, and their A-spectrum (resp., L-spectrum) are investigated. For a graph G, let $R(G)$ be the graph obtained from G by adding a new vertex u_e and joining u_e to the end vertices of e for each $e \in E(G)$. The graph $R(G)$ appeared in [11] and we call it the R-graph of G. Let $I(G)$ be the set of newly added vertices, i.e $I(G) = V(R(G)) \backslash V(G)$.

Let G_1 and G_2 be two vertex-disjoint graphs.

Definition 1.1 (*Lan and Zhou [10]*). The R-vertex corona of G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the ith vertex of $V(G_1)$ to every vertex in the ith copy of G_2.

Definition 1.2 (*Lan and Zhou [10]*). The R-edge corona of G_1 and G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the ith vertex of $|I(G_1)|$ to every vertex in the ith copy of G_2.

Note that if G_i has n_i vertices and m_i edges for $i = 1,2$, then $G_1 \odot G_2$ has $n_1 + m_1 + n_1 n_2$ vertices and $3m_1 + n_1 m_2 + n_1 n_2$ edges, $G_1 \ominus G_2$ has $n_1 + m_1 + m_1 n_2$ vertices and $3m_1 + m_1 m_2 + m_1 n_2$ edges.

As the authors of [12] pointed out, it is an interesting problem to compute Kirchhoff index of large composition graphs in terms of parameters of small graph in the composition [13,14]. The Kirchhoff index has been computed for some classes of graphs, such as cycles [15], complete graph [15], distance transitive graphs [16], and so on [5,8,15,17,20–25,31]. The Kirchhoff index of certain composite operations between two graphs was studied as well, such as product, lexicographic product [18] and join, corona, cluster [12]. Then recently Liu et al. [19] explore the Laplacian polynomial of $RT(G)$ of a regular graph G. Motivated by these results, in this paper we compute the Laplacian polynomial of $G_1 \odot G_2$ and $G_1 \ominus G_2$ for a regular graph G_1 and an arbitrary graph G_2 and derive formulae and low bounds of Kirchhoff index of these graphs and generalize their results in [19].

2. Preliminaries

In this section, we determine the characteristic polynomials of graphs with the help of the *coronal* of a matrix. The $M-$*coronal* $T_M(\lambda)$ of an $n \times n$ matrix M is defined [26,27] to be the sum of the entries of the matrix $(\lambda I_n - M)^{-1}$, that is

$$T_M(\lambda) = \mathbf{1_n^T}(\lambda \mathbf{I_n} - \mathbf{M})^{-1}\mathbf{1_n},$$

where $\mathbf{1}_n$ denotes the column vector of dimension n with all the entries equal one.

If M has a constant row sum t, it is easy to verify that

$$T_M(\lambda) = \frac{n}{\lambda - t}. \tag{1}$$

The Kronecker product $A \otimes B$ of two matrices $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{p \times q}$ is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. This is an associate operation with the property that $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$ whenever the products AC and BD exist.

Lemma 2.1 (*Zhang [28]*). *Let M_1, M_2, M_3 and M_4 be respectively $p \times p, p \times q, q \times p$ and $q \times q$ matrices with M_1 and M_4 invertible, then*

$$\det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_4)\det(M_1 - M_2 M_4^{-1} M_3) = \det(M_1)\det(M_4 - M_3 M_4^{-1} M_2),$$

where $M_1 - M_2 M_4^{-1} M_3$ and $M_4 - M_3 M_1^{-1} M_2$ are called the Schur complements of M_4 and M_1, respectively.

3. The Laplacian polynomial of *R*-vertex and *R*-edge corona

For a regular graph G_1, the next theorems give the representation of the Laplacian polynomial of $G_1 \odot G_2$ and $G_1 \ominus G_2$ by means of the characteristic polynomial and the Laplacian polynomial of G_1 and G_2.

Theorem 3.1. *Let G_1 be an r_1-regular graph with n_1 vertices and m_1 edges and G_2 be an arbitrary graph with n_2 vertices. Then the Laplacian characteristic polynomial of $G_1 \odot G_2$ is given by*

(i)
$$\mu_{G_1 \odot G_2}(x) = \prod_{i=1}^{n_2} (x - 1 - \mu_i(G_2))^{n_1}(x-2)^{m_1}(3-x)^{n_1} P_{G_1}\left(\frac{(x-n_2)(x-2)}{3-x} + \frac{r_1(2x-3)}{x-3} + \frac{n_2(x-2)}{(x-3)(x-1)}\right).$$
(ii)
$$\mu_{G_1 \odot G_2}(x) = \prod_{i=1}^{n_2} (x - 1 - \mu_i(G_2))^{n_1}(x-2)^{m_1}(x-3)^{n_1} \mu_{G_1}\left(\frac{x(x^2 - (3 + n_2 + r_1)x + (2n_2 + r_1 + 2))}{(x-1)(x-3)}\right).$$

Proof. (i) Let B be the vertex-edge incidence matrix of G_1. Since G_1 is an r_1-regular graph, we have $D(G_1) = r_1 I_{n_1}$. By a pertinent labeling of the vertices of $G_1 \odot G_2$, then the Laplacian matrix of $G_1 \odot G_2$ can be written as

$$L(G_1 \odot G_2) = \begin{pmatrix} 2I_{m_1} & -B^T & 0_{m_1 \times n_1 n_2} \\ -B & (r_1 + n_2)I_{n_1} + L(G_1) & -I_{n_1} \otimes 1_{n_2}^T \\ 0_{n_1 n_2 \times m_1} & -I_{n_1} \otimes 1_{n_2} & I_{n_1} \otimes (I_{n_2} + L(G_2)) \end{pmatrix},$$

where 0_n denotes the length-n column vectors consisting entirely of $0'$s.

It follows that

$$\mu_{G_1 \odot G_2}(x) = \det\begin{pmatrix} (x-2)I_{m_1} & B^T & 0_{m_1 \times n_1 n_2} \\ B & (x - r_1 - n_2)I_{n_1} - L(G_1) & I_{n_1} \otimes 1_{n_2}^T \\ 0_{n_1 n_2 \times m_1} & I_{n_1} \otimes 1_{n_2} & I_{n_1} \otimes ((x-1)I_{n_2} - L(G_2)) \end{pmatrix} = \prod_{i=1}^{n_2} (x - 1 - \mu_i(G_2))^{n_1} \cdot \det(S), \tag{2}$$

where

$$S = \begin{pmatrix} (x-2)I_{m_1} & B^T \\ B & (x - r_1 - n_2)I_{n_1} - L(G_1) \end{pmatrix} - \begin{pmatrix} 0_{m_1 \times n_1 n_2} \\ I_{n_1} \otimes 1_{n_2}^T \end{pmatrix}(I_{n_1} \otimes ((x-1)\, I_{n_2} - L(G_2))^{-1}\begin{pmatrix} 0_{n_1 n_2 \times m_1} & I_{n_1} \otimes 1_{n_2} \end{pmatrix}$$
$$= \begin{pmatrix} (x-2)I_{m_1} & B^T \\ B & (x - r_1 - n_2)I_{n_1} - L(G_1) \end{pmatrix} - \begin{pmatrix} 0_{m_1 \times m_1} & 0_{m_1 \times n_1} \\ 0_{n_1 \times m_1} & I_{n_1} \otimes 1_{n_2}^T (xI_{n_2} - L(G_2))^{-1} 1_{n_2} \end{pmatrix}$$
$$= \begin{pmatrix} (x-2)I_{m_1} & B^T \\ B & (x - r_1 - n_2)I_{n_1} - L(G_1) \end{pmatrix} - \begin{pmatrix} 0_{m_1 \times m_1} & 0_{m_1 \times n_1} \\ 0_{n_1 \times m_1} & T_{L(G_2)}(x) I_{n_1} \end{pmatrix}$$
$$= \begin{pmatrix} (x-2)I_{m_1} & B^T \\ B & (x - r_1 - n_2 - T_{L(G_2)}(x))I_{n_1} - L(G_1) \end{pmatrix}.$$

From (1), we have $T_{L(G_2)}(x-1) = \frac{n_2}{x-1}$ as each row sum of $L(G_2)$ is equal to 0. It is well known that $BB^T = A(G_1) + r_1 I_{n_1}$. Consequently,

$$\det(S) = \det((x-2)I_{m_1})\det\left(\left(x - r_1 - n_2 - \frac{n_2}{x-1}\right)I_{n_1} - L(G_1) - \frac{1}{x-2}BB^T\right)$$
$$= (x-2)^{m_1}\det\left(\left(x - 2r_1 - n_2 - \frac{n_2}{x-1} - \frac{r_1}{x-2}\right)I_{n_1} + \left(1 - \frac{1}{x-2}\right)A(G_1)\right)$$
$$= (x-2)^{m_1}(3-x)^{n_1}\det\left(\left(\frac{(x-n_2)(x-2)}{3-x} + \frac{r_1(2x-3)}{x-3} + \frac{n_2(x-2)}{(x-3)(x-1)}\right)I_{n_1}\right.$$