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Letters

On stability of disturbed Hopfield neural networks with time delays

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Abstract

In real application, the dynamics of Hopfield neural network is often affected by disturbing signals and time delays, so it is worthwhile to study dynamical properties of this type of neural network. Firstly, the ideal solution is defined as the solution of the network without disturbing signals. In order to ensure uniqueness, L_2-gain stability, global stability or global exponential stability of the ideal solution, corresponding sufficient conditions are presented, respectively, using homotopic method, inequality techniques, M-matrix properties or one time-delay inequality. All the obtained results are illustrated by several simulations.

Keywords: Hopfield neural network; Ideal solution; L_2-gain stability; Global stability; Exponential stability

1. Introduction

Hopfield neural network (HNN) has been widely used in many applications, such as solving the traveling salesman problem, etc. So its dynamics has been widely and deeply studied including global convergence [10], exponential stability [9,14,17], global asymptotic or exponential stability [1–3,6–8,12,14,16,18–21,23], instability [5,14], equilibria [4] and multistability [11]. The generalized form of HNN which has been studied involves assuming the network has graded response [17], time delays [16], time-varying [2] or distributed delays [19], parameters bounded in a compact set [6], or assuming the activation functions occupy multilevel trait [11]. In this paper, we study a generalized form of HNN which contains disturbance quantities and time delays, the model is described by the following state equations:

$$C_i\dot{u}_i(t) = -u_i(t)/R_i + \sum_{j=1}^{n} W_{ij}g(u_j(t-\tau_j) + \phi_j(t)) + \hat{\varphi}_i(t) + \hat{\theta}_i,$$
$$t-\tau_i \geqslant 0,\quad i=1,\ldots,n, \tag{1}$$

where $C_i>0$, $R_i>0$ and $\hat{\theta}_i>0$ are capacity, resistance, and external input, respectively. W_{ij} denotes the connection weight from neuron i to neuron j. $\phi_j(t), \widehat{\varphi}_i(t)$ denote internal, and external, disturbing signals, respectively, time delay $\tau_j \in [0, \vartheta]$, $\vartheta > 0$ is a constant. The activation function $g(\cdot)$ is an increasing and Lipschitz continuous function.

Let $A = \mathrm{diag}(a_1,\ldots,a_n) = \mathrm{diag}((R_1C_1)^{-1},\ldots,(R_nC_n)^{-1})$, $T = (W_{ij}/C_i)_{n\times n}, u(t-\tau) = (u_i(t-\tau_i))_{n\times 1}$, $g(u(t-\tau)) = (g(u_i(t-\tau_i)))_{n\times 1}$, $\phi(t) = (\phi_i(t))_{n\times 1}$, $\varphi(t) = (\hat{\varphi}_i(t)C_i^{-1})_{n\times 1}$ and $\theta = (\hat{\theta}_iC_i^{-1})_{n\times 1}$. When disturbing signals vanish, network (1) equals to

$$\dot{u}(t) = -Au(t) + Tg(u(t-\tau)) + \theta. \tag{2}$$

Let u_i^* be a solution of formula (2), $x_i(t) = u_i(t) - u_i^*$ and $f(u_i(t-t_i) + \phi_i(t)) = g(u_i(t-t_i) + \phi_i(t)) - g(u_i^*)$, formula (1) can be written as

$$C_i\dot{x}_i(t) = -x_i(t)/R_i + \sum_{j=1}^{n} W_{ij}f(u_j(t-\tau_j) + \phi_j(t)) + \hat{\varphi}_i(t). \tag{3}$$

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Obviously, the trivial solution of formula (3) is corresponding to the ideal solution of network (1), therefore analyzing the stability of the trivial solution of formula (3) is equal to analyzing that of the ideal solution of formula (1).

This paper is organized as follows: some preliminaries are given in Section 2, Section 3 introduces a sufficient condition ensuring uniqueness of the ideal solution. Section 4 presents some sufficient conditions ensuring L_2-gain stability, global stability or global exponential stability of the ideal solution. Some simulations and discussions, final conclusions are given in Sections 5 and 6, respectively.

2. Some preliminaries

Let I denote a suitable identity matrix and $x(t) = (x_i(t))_{n\times 1}$, and define

$$||x(t)|| = \sqrt{\sum_{i=1}^{n}\int_0^{\infty} ||x_i(t)||^2\,\mathrm{d}t}.$$

Definition 1. For a system $f: R^n \times R^n \to R^n$, $\dot{x} = f(x,u)$, $x, u \in R^n$, if the solution of this system $x(t) \in L_2([0,\infty), R^n)$ satisfies $||x|| \leqslant M||u||$, then the minimal value of M denoted by $\mathrm{Min}(M)$ is called the L_2-gain of this system. If $\mathrm{Min}(M)$ is bounded, the system is called L_2-gain stable.

Definition 2 (*Arik [1]*). A matrix $H = (h_{ij})_{n\times n}$ is said to be diagonally stable if there exists a positive diagonal matrix P such that $H^{\mathrm{T}}P + PH$ is positive definite. If $\alpha I + H$ is diagonally stable for $\alpha > 0$, then H is said to be additively diagonally stable.

Definition 3 (*Guan et al. [5]*). H is said to an M-matrix if there exists a vector x (or y) whose elements are all positive such that the elements of Hx (or $H^{\mathrm{T}}y$) are all positive.

Lemma 1 (*Zhang et al. [22]*). *Assume* $S_i(t) \in \lfloor [t_0 - \tau, +\infty), R^+ \rfloor$ *and satisfies*

$$\dot{S}_i(t) \leqslant -r_i S_i(t) + h_i(S_i)S_t \quad (i = 1,\ldots,n),$$

where $r_i > 0$, $h_i(\cdot)$ *is a non-negative, non-decreasing and continuous function*,

$$S_t \equiv \max_{1\leqslant j\leqslant n}\left[\sup_{t-\tau\leqslant\theta\leqslant t}(S_j(\theta))\right].$$

If there exists a constant $k > 0$ *satisfying* $h_i(k) < r_i$ $(i = 1,\ldots,n)$, *then there has*

$$S_i(t) \leqslant S_{t_0}\cdot \mathrm{e}^{-\varepsilon(t-t_0)}, \quad t \geqslant t_0, \ \varepsilon \in R^+ \ \textit{when}$$
$$S_{t_0} < k \quad (i = 1,\ldots,n).$$

3. Uniqueness of the ideal solution

Theorem 1. *If* $-T$ *is additively diagonally stable, the ideal solution of formula* (1) *is unique, i.e. formula* (2) *has only one solution.*

Proof. Assume formula (2) has at least two solutions, without any loss of generality, u^{*1} and u^{*2} denote any two of them. Obviously, we have

$$Au^{*1} - Tg(u^{*1}) - \theta = 0, \tag{4}$$

$$Au^{*2} - Tg(u^{*2}) - \theta = 0. \tag{5}$$

When the equilibrium state u^{*2} is destroyed, there are two cases: (1) System (2) returns to state u^{*2}, in this instance, system (2) has only one equilibrium state u^{*2} since it is randomly selected, Theorem 1 is naturally tenable. (2) System migrates to other equilibrium state, since u^{*1} is randomly selected, assume system migrates to u^{*1}. In this case, the state trajectory of migration can be written as

$$\begin{aligned} u(t) &\equiv \lambda(t)u^{*1} + (I - \lambda(t))u^{*2} \\ &= u^{*2} + \lambda(t)(t)(u^{*1} - u^{*2}), \end{aligned} \tag{6}$$

where $\lambda(t) = \mathrm{diag}(\lambda_i(t))$. Assuming $u(t_0) = u^{*2}$ and $u(t_1) = u^{*1}$, we have

$$\lambda_i(t_0) = 0, \quad \lambda_i(t_1) = 1 \tag{7}$$

and

$$\dot{\lambda}_i(t) > 0 \quad \text{when } t \in (t_0, t_1). \tag{8}$$

Substituting formula (6) into formula (2) gives that

$$\begin{aligned} \dot{\lambda}(t)(u^{*1} - u^{*2}) = {} & -A(u^{*2} + \lambda(t)(u^{*1} - u^{*2})) \\ & + Tg(u^{*2} + \lambda(t-\tau)(u^{*1} - u^{*2})) + \theta. \end{aligned} \tag{9}$$

Using formula (5) and (9) gives that

$$\begin{aligned} \dot{\lambda}(t)(u^{*1} - u^{*2}) = {} & -A\lambda(t)(u^{*1} - u^{*2}) \\ & + T(g(u^{*2} + \lambda(t-\tau)(u^{*1} - u^{*2})) \\ & - g(u^{*2})). \end{aligned} \tag{10}$$

Since $g(.)$ has increasing and Lipschitz continuous properties, so we have

$$\begin{aligned} & g(u^{*2} + \lambda(t-\tau)(u^{*1} - u^{*2})) - g(u^{*2}) \\ & \quad = k(t)\lambda(t-\tau)(u^{*1} - u^{*2}), \end{aligned} \tag{11}$$

where $k(t) = \mathrm{diag}(k_i(t))_{n\times n}$, $k_i(t) > 0$ and $\lambda(t-\tau) = \mathrm{diag}(\lambda_i(t-\tau_i))$. Substituting formula (11) into formula (10) gives that

$$\dot{\lambda}(t)(u^{*1} - u^{*2}) = -[A\lambda(t) - Tk(t)\lambda(t-\tau)](u^{*1} - u^{*2}). \tag{12}$$

Assume there exist a time $\widetilde{t} \in [t_0, t_1]$ satisfying $t_1 > \widetilde{t} - \vartheta > t_0$. Denote $\widetilde{u}(t) = k(t)\lambda(t-\tau)(u^{*1} - u^{*2})$. Since $\lambda_i(\widetilde{t} - \tau_i) > 0$, with any matrix $p = \mathrm{diag}(p_i > 0)$, we have

$$\widetilde{u}^{\mathrm{T}}(\widetilde{t})p\dot{\lambda}(\widetilde{t})[k(\widetilde{t})\lambda(\widetilde{t} - \tau)]^{-1}\widetilde{u}(\widetilde{t}) > 0. \tag{13}$$

Since $-T$ is additively diagonally stable, using Definition 2 gives that

$$\widetilde{u}^{\mathrm{T}}p\{A\lambda(t)[k(t)\lambda(t-\tau)]^{-1} - T\}\widetilde{u}(t) \geqslant 0. \tag{14}$$

Substituting formula (13) and (14) into (12) gives that formula (12) is a contradictive relation, this contraction implies that there are not two equilibrium states for system (2) when $-T$ is additively diagonally stable, i.e., the ideal