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# New delay-dependent stability criteria for recurrent neural networks with time-varying delays

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ABSTRACT

This work is concerned with the delay-dependentstability problem for recurrent neural networks with time-varying delays. A new improved delay-dependent stability criterion expressed in terms of linear matrix inequalities is derived by constructing a dedicated Lyapunov–Krasovskii functional via utilizing Wirtinger inequality and convex combination approach. Moreover, a further improved delay-dependent stability criterion is established by means of a new partitioning method for bounding conditions on the activation function and certain new activation function conditions presented. Finally, the application of these novel results to an illustrative example from the literature has been investigated and their effectiveness is shown via comparison with the existing recent ones.



## 1. Introduction

During the past several decades, an increasing, revived research activity on recurrent neural networks (RNNs) is taking place because of their successful applications in various areas. These include associative memories, image processing, optimization problems, and pattern recognition as well as other engineering or scientific areas [1–5]. It is well known, the time delay often is a source of the degradation of performance and/or the instability of RNNs. It is therefore that the stability analysis of RNNs with time delays has attracted considerable attention in recent years, e.g. see Refs. [6–11] and references therein.

It should be noted, the existing stability criteria for RNNs with time delays can be classified into the delay-independent ones and the delay-dependent criteria. In general, when the time delay is small, the delay-dependent stability criteria are less conservative than delay-independent ones. For the delay-dependent stability criteria, the maximum delay bound is a very important index for checking the criterion's conservatism. In due course, significant research efforts have been devoted to the reduction of conservatism of the delay-dependent stability criteria for the time-delay RNNs. Following the Lyapunov stability theory, there are two effective ways to reduce the conservatism within in stability analysis of networks and systems. One is the choice of suitable Lyapunov–Krasovskii functional (LKF) and the other one is the estimation of its time derivative.

In recent years, some new techniques of construction of a suitable LKF and estimation of its derivative for delayed neural networks (DNNS) and time delay systems have been presented [12–32,44–48]. Methods for constructing a dedicated LKF include delay-partitioning idea [12–20], triple integral terms [16–25], more information on the activation functions [26], augmented vector [27,28], etc. The proposed methods for estimating the time-derivative of LKF include: Park' inequality [29], Jensen's inequality [30], free-weighing matrices [31], and reciprocally convex optimization [32]. In turn, these methods proved very useful in investigating the stability problems of RNNs with time delays. Among the stability analysis methods, some delay-dependent criteria for the RNNs with time-varying delays have been contributed in works [33–36,42]. For instance, in Ref. [33] the problem of delay-dependent stability has been investigated by considering some semi-positive-definite free matrices. Jensen's inequality combined with convex combination method has been used in Ref. [35]. In Ref. [36] a new improved delay-dependent stability criterion was proposed, which has been derived by constructing a new augmented LKF, containing a triple integral term, also by using Wirtinger-based

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integral inequality and two zero value free matrix equations. However, the introduced free-weighing matrices increase the calculation complexity as well as computational complexity. For the RNNs with interval time-varying delays, work [43] has contributed an improved stability criterion by construction of a suitable augmented LKF and utilization of Wirtinger-based integral inequality with reciprocally convex approach. Following the work [37], both the ability and the performance of neural networks are influenced considerably by the choice of the activation functions. Apparently there is an essential need to look for alternative methods of reducing the conservatism of stability criteria for such neural networks. Thus, the delay-partitioning approach appeared as an effective way to get a tighter bound by calculating the derivative of the LKF, which would lead to better results. However, as the partitioning number of delay increases, the matrix formulation becomes more complex and the dimensionality of the stability criterion grows bigger. Hence the computational burden and computational time consumption growth become a considerable problem. The activation function dividing approach was proposed in work [23], and some new improved delay-dependent criteria for neural networks with time-varying delays have been established. A more general activation function dividing method for delay-dependent stability analysis of DNNs was presented in Ref. [38].

The above motivating discussion has given considerably incentives to utilize a modified approach, albeit making use of the existing knowledge, in order to arrive at less conservative, novel, delay-dependent stability criteria for recurrent neural networks with time-varying delays.

Firstly, a combined convex method is developed for the stability of the recurrent neural network systems with time-varying delays. This method can tackle both the presence of time-varying delays and the variation of delays. As a first novelty, a new LKF is constructed by taking more information on the state and the activation functions as augmented vectors. It has been found by using reciprocal convex approach and Wirtinger inequality to handle the integral term of quadratic quantities. With the new LKF at hand, in Theorem 1, the delay-dependent stability criterion in which both the upper and lower bounds of delay derivative are available is then derived. Secondly, unlike the delay partitioning method, a new dividing approach of the bounding conditions on activation function is utilized in Theorem 2. Considering the time and the improvement of the feasible region, the bounding of activation functions $k_i^- \le (f_i(u)/u) \le k_i^+$ of RNNs with time-varying delays is divided into two subintervals such as to obtain: $k_i^- \le (f_i(u)/u) \le k_i^- + \alpha(k_i^+ - k_i^-)$ and $k_i^- + \alpha(k_i^+ - k_i^-) \le (f_i(u)/u) \le k_i^+$ $(0 \le \alpha \le 1)$, where the two sub-intervals can be either equal or unequal. New activation function conditions for the divided activation functions bounds are proposed and utilized in Theorem 2. Thirdly, by utilizing the results of Theorems 1 and 2, when only the upper bound of the derivative of the time-varying delay is available, the corresponding new results are proposed in Corollaries 1 and 2. Finally, this stability analysis method was applied to a known example from the literature and the respective results computed. These new results were compared with the existing recent ones in order to verify and illustrate the effectiveness of the new method and to demonstrate the improvements obtained. Further, Section 2 presents the problem formulation and Section 3 presents the new main results. Section 4 elaborates on the illustrative example and comparison analysis, while conclusions are drawn and further research outlined in Section 5.

This paper uses the following notations: $C^T$ represents the transposition of matrix $C$. $\mathbb{R}^n$ denotes $n$-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. $P > 0$ means that $P$ is positive definite. Symbol $*$ represents the elements below the main diagonal of a symmetric block matrix, and $diag\{\cdots\}$ denotes a block diagonal matrix. $Sym(X)$ is defined as $Sym(X) = X + X^T$.

## 2. Problem formulation

Consider the following recurrent neural networks with discrete time-varying delays:

$$\dot{z}(t) = -Az(t) + f(Wz(t-h(t)) + J) \tag{1}$$

where $z(\cdot) = [z_1(\cdot), ..., z_n(\cdot)]^T$ is the state vector; $f(\cdot) = [f_1(\cdot), ..., f_n(\cdot)]^T$ denote the neuron activation functions; $J = [J_1, ..., J_n]^T \in \mathbb{R}^n$ is a vector representing the bias; $A = diag\{a_1, ..., a_n\} \in \mathbb{R}^{n\times n}$ is a constant matrix of appropriate dimensions; $W = [W_1, ..., W_n]^T \in \mathbb{R}^n$ represents the matrix of connection weights; and $h(t)$ is a time-varying delay having the following bound properties

$$C1: \quad 0 \le h(t) \le h, \ h_D^l \le \dot{h}(t) \le h_D^u < 1,$$

$$C2: \quad 0 \le h(t) \le h, \ \dot{h}(t) \le h_D^u.$$

where $h > 0$ and $h_D^l$, $h_D^u$ are known constants.

The activation functions $f_i(\cdot)$, $i = 1, ..., n$ are assumed to be bounded and to satisfy the following bound conditions:

$$k_i^- \le \frac{f_i(u) - f_i(v)}{u - v} \le k_i^+, \quad u \ne v, \ i = 1, ..., n \tag{2}$$

where $k_i^-$ and $k_i^+$ are constants.

In the stability analysis of recurrent neural networks (1), for simplicity, firstly we shift the equilibrium point $z^*$ to the origin by letting $x = z - z^*$. Then the system (1) can be converted into

$$\dot{x}(t) = -Ax(t) + g(Wx(t-h(t))) \tag{3}$$

where $g(\cdot) = [g_1(\cdot), ..., g_n(\cdot)]^T$ and $g(Wx(\cdot)) = f(Wx(\cdot) + z^* + J) - f(Wz^* + J)$ with $g_i(0) = 0$. Notice that functions $g_i(\cdot)$ $(i = 1, ..., n)$ satisfy the following bound conditions:

$$k_i^- \le \frac{g_i(u) - g_i(v)}{u - v} \le k_i^+, \quad u \ne v, \ i = 1, ..., n. \tag{4}$$

If $v = 0$ in (4), then these inequalities become

$$k_i^- \le \frac{g_i(u)}{u} \le k_i^+, \quad \forall u \ne 0, \ i = 1, ..., n. \tag{5}$$

The objective of this paper is to explore of asymptotic stability of recurrent neural networks (3) with time-varying delays and to establish a novel analysis method. Before deriving the main results of this contribution, the following lemmas are needed:

**Lemma 1.** *[32,39]Consider the given positive integers n, m, a positive scalar $\alpha$ in the interval $(0,1)$, a given $n \times n$ matrix $R > 0$, two matrices $W_1$ and $W_2$ in $\mathbb{R}^{n\times m}$. For all vectors $\xi$ in $\mathbb{R}^m$, define the function $\Theta(\alpha, R)$ as*

$$\Theta(\alpha, R) = \frac{1}{\alpha}\xi^T W_1^T R W_1 \xi + \frac{1}{1-\alpha}\xi^T W_2^T R W_2 \xi.$$

Then, if there exists a matrix $X$ in $\mathbb{R}^{n\times n}$ such that $\begin{bmatrix} R & X \\ * & R \end{bmatrix} > 0$, the following inequality holds true:

$$\min_{\alpha \in (0,1)} \Theta(\alpha, R) \ge \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}.$$

**Lemma 2.** *[39] For a given matrix $R > 0$, the following inequality holds for all continuously differentiable functions $\sigma$ in $[a,b] \to \mathbb{R}^n$:*

$$\int_a^b \dot{\sigma}^T(u) R \dot{\sigma}(u) du \ge \frac{1}{b-a}(\sigma(b) - \sigma(a))^T R (\sigma(b) - \sigma(a)) + \frac{3}{b-a}\delta^T R \delta,$$

*where $\delta = \sigma(b) + \sigma(a) - (2/b-a)\int_a^b \sigma(u) du$.*

**Lemma 3.** *[40] Let $\xi \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{m\times n}$ such that rank $(B) < n$. Then, the following statements are equivalent:*

(1) $\xi^T \Phi \xi < 0$, $B\xi = 0$, $\xi \ne 0$,
(2) $(B^\perp)^T \Phi B^\perp < 0$, *where $B^\perp$ is a right orthogonal complement of B.*