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Finite time stability of periodic solution for Hopfield neural networks with discontinuous activations ☆

Xiaoyan Chen [a], Lihong Huang [a,b,*], Zhenyuan Guo [a]

[a] College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China
[b] Hunan Women's University, Changsha, Hunan 410004, PR China

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ABSTRACT

Based on the tangency or non-tangency of the periodic solution to certain surface, this paper gives a set of conditions ensuring global convergence in finite time toward a unique periodic solution for Hopfield neural networks with discontinuous activations. Moreover, two numerical examples are provided to illustrate the theoretical results.

1. Introduction

In recent years, considerable efforts have been devoted to studying neural networks which can be applied to various science and engineer fields such as image and signal processing, optimization, learning memory, and so on (see [1–4]). Such applications closely relate to the dynamical behaviors of designed neural networks, such as stability, periodical oscillatory, bifurcation and chaos. As shown by [5], in many applications, periodic oscillatory is common and the property of periodic oscillatory solutions is interesting. For example, the human brain is in periodic oscillatory or chaos state. In addition, when the neural networks model the learning memory, periodic solution of the designed system means that obtaining the idea needs activities or motion repetitions. Hence, it is important to study neural networks with periodic oscillatory.

However, up to now, most of the results concerning the neural networks are based on the assumption that the activations are continuous or even Lipschitzian (see [1–7] and references therein). To the best of our knowledge, the paper [8] written by Forti and Nistri is the first one to handle the global stability of neural networks with discontinuous activations. As it pointed out, neural networks with discontinuous activations are of importance and frequently arise in practice. For example, modeling the dynamical systems with high slope nonlinearity, the discontinuous differential equations may be better candidates than continuous ones with high and finite slope. So far, the results in Ref. [8] have been extended by many scholars (see [9–16]).

It is worth to note that the finite time stability of equilibrium point has been considered in Refs. [9–12]. The global convergence in finite time toward equilibrium point means all the solutions reach equilibrium point in finite time not just asymptotically approach it in infinite time. According to the dynamical qualitative theory, to achieve convergence in finite time, the differential systems need to be non-Lipschitzian, as non-Lipschitzian property can give rise to non-uniqueness of solutions in reverse time. Due to the discontinuity of right hand sides, many discontinuous differential systems present the property of finite time stability. In addition, the analysis of the discontinuous cases might reveal some other important traits of dynamics, such as the presence of sliding modes along the discontinuity surfaces and the ability of computing the exact global minimum of the underlying energy functions, which have been applied in the design of real-time control and optimization solver (see [17–22]). As far as we know, most of the existing literatures involving the global convergence in finite time are concerning equilibrium point (see [17–24]).

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* Corresponding author at: College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China.
E-mail addresses: xychen@hnu.edu.cn (X. Chen), lhhuang@hnu.edu.cn (L. Huang).

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However, as noted by Liu [25], the phenomenon of converging toward the limit cycle in finite time is also an important dynamical trait. Although there have been some papers (see [14,26]) studying the finite time stability of periodic solution, no one presents rigorous theoretical proof. For example, Ref. [26] only gives a numerical example where periodic solution is finite time stable without theoretical proof. And the authors of [14] have shown the proof of the finite time stability of periodic solution of bi-directional associative memory neural networks, but it is wrong.

Motivated by the above analysis, we take the following Hopfield neural networks as an example to present rigorous theoretical proof of the finite time stability of periodic solution:

$$x'(t)=f(t,x(t))=-D(t)x(t)+A(t)g(x(t))+I(t), \tag{1.1}$$

where $x(t)=(x_1(t),\dots,x_n(t))^T\in\mathbb{R}^n$ is the vector of neuron states; $D(t)=\mathrm{diag}(d_1(t),\dots,d_n(t))\in\mathbb{R}^{n\times n}$ is an ω-periodic diagonal matrix function with diagonal entries $d_i(t)>0$, $i=1,\dots,n$, denoting the neuron self-inhibition; $A(t)=(a_{ij}(t))_{n\times n}$ is an ω-periodic matrix function whose entries denote the neuron interconnections; $I(t)=(I_1(t),\dots,I_n(t))^T$ is an ω-periodic vector function representing the neuron input; $g(x)=(g_1(x_1),\dots,g_n(x_n))^T$, where g_i, $i=1,\dots,n$, are nonlinear neuron input–output activations and for each $i=1,\dots,n$, g_i satisfies the following conditions:

(C1) g_i is piecewise continuous, i.e., g_i is continuous in $\mathbb{R}$ except a countable set $M_i=\{\rho_{i1},\dots,\rho_{ik_i},\dots\}$, where $\rho_{i1}<\rho_{i2}<\cdots<\rho_{ik_i}<\cdots$ and for each point $\rho_{ij}\in M_i$, there exist finite right limit and left limit, $g_i((\rho_{ij})^+)$ and $g_i((\rho_{ij})^-)$, respectively and $g_i((\rho_{ij})^+)-g_i((\rho_{ij})^-)>0$; moreover, g_i has at most finite number of discontinuous points on any compact interval of $\mathbb{R}$.

(C2) g_i is monotone nondecreasing.

(C3) $D(t)$, $A(t)$, $I(t)$ are measurable and locally bounded.

Some conditions have been given to ensure the existence, uniqueness and asymptotical stability of periodic solution for system (1.1) in Ref. [15]. Based on the existent results, we further study the finite time stability of periodic solution for system (1.1).

The rest of this paper is organized as follows. Some preliminaries concerning Filippov theory are presented in Section 2. Section 3 shows the main theorems on finite time stability of periodic solution. Two numerical examples are given to illustrate the results in Section 4. At last, we state the conclusion in Section 5.

2. Preliminaries

In this section, we present some knowledge concerning Filippov theory which will be used throughout this paper. For more details, the readers can refer to [27,28].

In this paper, we consider the Filippov solution of system (1.1), which is defined as follows:

Definition 2.1 (*See Papini and Taddei [15]*). A function $x:[t_0,T)\longrightarrow\mathbb{R}^n$ is a Filippov solution of system (1.1) if

(1) x is absolutely continuous on any compact interval $I\subset[t_0,T)$;

(2) $x'(t)\in F(t,x)=-D(t)x(t)+A(t)K[g(x(t))]+I(t)$ for a.a. $t\in[t_0,T)$, (2.1)

where $K[g(x(t))]=(K[g_1(x_1(t))],\dots,K[g_n(x_n(t))])^T$ and for $i=1,\dots,n$,
$K[g_i(x_i)]=[g_i(x_i^-),g_i(x_i^+)]$.

Obviously, $F(t,x)$ is nonempty bounded closed convex valued and upper semi-continuous in x [27, p. 67, Lemma 1]. The solution exists in any compact interval [27, p. 83 Theorem 5]. What is more, the upper semi-continuity implies measurability [27, p. 72]. By the measurable selection theorem [29, p. 308], if $x(t)$ is a solution of system (2.1), there exists a bounded measurable function $\gamma(t)\in K[g(x(t)]$ such that

$$x'(t)=-D(t)x(t)+A(t)\gamma(t)+I(t)\quad\text{for a.a. } t\in[t_0,T). \tag{2.2}$$

Function $\gamma(t)$ represents the neural network output on $[t_0,T)$. Moreover, $[x^*(t),\gamma^*(t)]$ is said to be an ω-periodic solution of system (2.1), if

$$x^*(t+\omega)=x^*(t)\text{ for all } t\ge t_0\quad\text{and}\quad \gamma^*(t+\omega)=\gamma^*(t)\text{ for a.a. } t\ge t_0.$$

Now, let us introduce the definition of right uniqueness which is a necessary condition for the global convergence in finite time toward the periodic solution.

Definition 2.2 (*See Filippov [27]*). For system (2.1), the right uniqueness holds at a point (t_0,x_0) if there exists $t_1>t_0$ such that each two solutions of system (2.1) satisfying the condition $x(t_0)=x_0$ coincide on the interval $[t_0,t_1]$.

For system (2.1), right uniqueness holds in a domain D if for each point $(t_0,x_0)\in D$ every two solutions satisfying the condition $x(t_0)=x_0$ coincide on each interval $[t_0,t_1]$ on which they both exist and lie in this domain.

Definition 2.3. The ω-periodic solution $x^*(t)$ of system (2.1) is finite time stable if for any $x_0\in\mathbb{R}^n$, $t_0\in\mathbb{R}$, there exist $t^*\in[0,\omega]$ and $T(t_0,x_0)<\infty$ such that all the solutions $x(t,t_0,x_0)$ of system (2.1) with the initial condition $x(t_0)=x_0$ satisfy

$$x(T(t_0,x_0),t_0,x_0)=x^*(t^*)\quad\text{and}\quad x(t+T(t_0,x_0))=x^*(t+t^*)\ \text{for } t\ge 0.$$

Lemma 2.1 (*Chain Rule, see Huang et al. [28]*). *Suppose that $V(t,x):\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is regular and that $x(t):[0,\infty)\longrightarrow\mathbb{R}^n$ is absolutely continuous on any compact interval of $[0,\infty)$. Then $x(t)$ and $V(t,x(t))$ are differential for a.a. $t\in[0,\infty)$, and we have*

$$\frac{d}{dt}V(t,x)=\frac{\partial V(t,x)}{\partial t}+\langle\xi,x'(t)\rangle,\quad\forall\xi\in\frac{\partial V(t,x)}{\partial x}.$$

3. Main results

In this section, two main theorems concerning the finite time stability of periodic solution are shown. First of all, we give some notations and assumptions.

Let

$$\overline{a}_{ij}=\mathrm{ess}\sup_{t\in[0,\omega]}|a_{ij}(t)|,\quad \underline{a}_{ii}=-\mathrm{ess}\sup_{t\in[0,\omega]}a_{ii}(t),\quad d_i=\mathrm{ess}\inf_{t\in[0,\omega]}d_i(t),$$

$$0<\lambda<\underline{d}=\min_{i=1,\dots,n}\{d_i\},\quad \|x\|_\xi=\sum_{i=1}^n\xi_i|x_i|,\quad \xi=(\xi_1,\dots,\xi_n)^T>0, \tag{3.1}$$

where $\xi=(\xi_1,\dots,\xi_n)^T>0$ means for each $i=1,\dots,n$, $\xi_i>0$ and for function $f:I\subset\mathbb{R}\longrightarrow\mathbb{R}$, $\mathrm{ess}\sup_{t\in I}f(t)=\min\{M\in\mathbb{R}:f(t)\le M$ for a.a. $t\in I\}$ and $\mathrm{ess}\inf_{t\in I}f(t)=\max\{M\in\mathbb{R}:f(t)\ge M$ for a.a. $t\in I\}$.

Assumption 3.1. For each $i=1,\dots,n$, we have $\underline{d}>0$ and the matrix $C=(c_{ij})_{n\times n}\in\mathbb{R}^{n\times n}$ given by

$$c_{ij}=\underline{a}_{ii}\text{ for } i=j;\quad c_{ij}=-\overline{a}_{ij}\text{ for } i\ne j,$$

is an M-matrix, i.e., C has all non-positive elements outside the diagonal and all positive successive principal minors.

Assumption 3.1 implies that there exists a positive vector $\beta=(\beta_1,\dots,\beta_n)^T$ such that $\beta^TC>0$ (see [30]).

According to Ref. [15], we can obtain the following two theorems.