Neurocomputing 151 (2015) 246–251

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Letters

# Finite-time solution to nonlinear equation using recurrent neural dynamics with a specially-constructed activation function

Lin Xiao *, Rongbo Lu

*College of Information Science and Engineering, Jishou University, Jishou 416000, China*

ARTICLE INFO

*Article history:*
Received 14 June 2014
Received in revised form
22 August 2014
Accepted 25 September 2014
Communicated by J. Zhang
Available online 8 October 2014

*Keywords:*
Activation function
Finite-time convergence
Nonlinear equation
Zhang dynamics

ABSTRACT

A recurrent neural dynamics (termed improved Zhang dynamics, IZD), together with a specially-constructed activation function, is proposed and investigated for finding the root of nonlinear equation in this paper. It is analyzed that the IZD model can converge to the theoretical solution within finite time. Besides, the upper bound of convergence time is estimated analytically. Compared with conventional gradient-based dynamics (GD), our proposed IZD model in the form of implicit dynamics has the following advantages: (1) has better consistency with actual situations; and (2) has a greater ability in representing dynamical systems. Besides, our model can achieve superior convergence performance (i.e., finite-time convergence) in comparison with the existing neural dynamics, specifically the original ZD (OZD) model. Both theoretical analysis and computer-simulation results substantiate the effectiveness and superiority of the IZD model for solving nonlinear equation in real-time.



## 1. Introduction

It is well known that a wide class of problems, which arises in engineering practices and scientific applications, can be studied via the nonlinear equations using the innovative techniques [1–5]. To solve the nonlinear-equation problem in real-time, in mathematics, almost all methods/techniques are based on the following defining equation [1–11]:

$$f(x) = 0 \in R, \tag{1}$$

where $f(\cdot): R \to R$ denotes a smooth nonlinear function. Due to the nonlinearity of $f(x)$, nonlinear equation (1) may have no solutions. Throughout the paper, we assume that the solution set of (1) is nonempty. For presentation convenience, let $x^*$ denote a theoretical solution (or termed, root) of nonlinear equation (1). It is worth pointing out that considerable attention has been focused to solve nonlinear equation (1) both analytically and numerically. For example, several iterative numerical methods have been developed using quite different techniques such as Taylor's series, quadrature formulas, homotopy, interpolation, decomposition and its various modification [6–11].

In recent years, due to the in-depth research in neural networks, various dynamic and analog solvers in the form of neural dynamics and recurrent neural networks have been developed, investigated and implemented on specific architectures [12–14]. In addition, as a software and hardware implementable approach, neural dynamics has widely arisen in scientific computation and optimization, drawing extensive interest and investigation of researchers [15–19]. Compared with conventional numerical algorithms, it has several potential advantages in real-time applications (e.g., parallel processing, distributed storage, self-adaptation, and high fault tolerance), and has been taken into account as one powerful computational scheme for online solution of nonlinear equation problems. However, most reported computational schemes are related to gradient-based methods and/or designed theoretically for solving time-invariant (or termed, static) problems [20–23]. Specifically, to solve nonlinear equation (1), we can design a classic gradient-based dynamics (GD) as follows [20–23].

Firstly, we can define a square-based energy function $\mathcal{E}(x) = f^2(x)/2$. Evidently, its minimum point can be achieved if and only if the solution $x$ of nonlinear equation (1) is equal to theoretical solution $x^*$. Secondly, a scheme can be designed to evolve along the negative gradient descent direction of the energy function until the minimum is reached. That is $-\partial\mathcal{E}(t)/\partial x = -f'(x)f(x)$. Finally, a typical learning rule based on the negative gradient generates the GD model as $\dot{x}(t) = -\gamma\partial\mathcal{E}(t)/\partial x = -\gamma f'(x)f(x)$, where design parameter $\gamma > 0$ is used to adjust the convergence rate of the GD model, and $x(t)$, starting from an initial state $x(0) \in R$, denotes the neural state corresponding to the theoretical solution $x^*$.

Recently, a special class of neural dynamics (termed Zhang dynamics, ZD) was originally proposed by Zhang et al. for solving online time-varying problems [20–26], which can also be applied to online solution of time-invariant problem. In [20,21], the original ZD (OZD) model has been generalized, developed, analyzed and

* Corresponding author.
*E-mail address:* xiaolin860728@163.com (L. Xiao).

http://dx.doi.org/10.1016/j.neucom.2014.09.047
0925-2312/

compared for online solution of nonlinear equations. Different from the GD model, the OZD has been elegantly introduced by defining an indefinite error-monitoring function. By comparing the GD model and the OZD model in [20,21], we can draw a conclusion that, for the situation of a simple root, the GD model and the OZD model can effectively converge to the theoretical simple root; and, for the situation of a multiple root, with the increase of the order of the multiple root, the neural state of the ZD model can still converge well to such a theoretical root; in contrast, the GD model more probably yields wrong (or approximate) solutions. In addition, the OZD model has been proven to converge to the theoretical solution ideally when time goes to infinity. However, the OZD model cannot converge to the desired value within finite time, which may limit its applications in real-time processing. Motivated by further improving the efficacy of the OZD model, an improved ZD (IZD) model is developed for online nonlinear equation solving. More importantly, the upper bound of the convergence time for the IZD model is derived analytically via the Lyapunov theory. To the best of authors' knowledge, this is the first time to provide such an IZD model for solving nonlinear equation problem and its finite-time neural solution. Before ending this section, it is worth summarizing and listing the main contributions and novelties of this paper as follows.

- For the first time, a novel finite-time convergent neural dynamics (i.e., the IZD model) together with a specially-constructed activation function is proposed and exploited for finding the root of nonlinear equation (1) in real-time.
- Our proposed IZD model can outperform the existing neural dynamical models, e.g., the conventional GD model and the recently-proposed OZD model, with superior convergence performance (i.e., finite-time convergence) achieved.
- Two illustrative examples are provided and the computer-simulation results can verify the effectiveness and superiority of our proposed IZD model for nonlinear equation solving in real-time.

## 2. Model formulation

To facilitate the convergence analysis and to lay a basis for further discussion, the following two dynamical models are first presented in this section.

### 2.1. Original Zhang dynamics (OZD)

To solving online nonlinear equation (1), following Zhang et al.' design method [24–26], we can develop a neural dynamics expressed as an implicit dynamic system (i.e., the original Zhang dynamics, OZD).

Firstly, to monitor the nonlinear-equation solving process, an indefinite error function $e(t)$ is defined below, instead of a square-based energy function associated with the aforementioned GD model:

$$e(t) = f(x). \tag{2}$$

Then, the ZD design formula is adopted such that $e(t)$ converges to zero as time $t$ goes on. That is

$$\frac{\mathrm{d}e(t)}{\mathrm{d}t} = -\gamma e(t). \tag{3}$$

Now, expanding the above design formula (3) and in view of $\dot{e}(t) = f'(x)\dot{x}(t)$, we can obtain the following implicit dynamic equation of the OZD model:

$$f'(x)\dot{x}(t) = -\gamma f(x), \tag{4}$$

where $x(t) \in R$, $f'(x)$ and $\gamma > 0 \in R$ are defined as the before.

### 2.2. Improved Zhang dynamics (IZD)

In this subsection, an improved Zhang dynamics (IZD) is proposed for finding the root of nonlinear equation (1) by adding a specially-constructed activation function [27–29] (i.e., the sign-bi-power function or the Li function):

$$f'(x)\dot{x}(t) = -\gamma \mathrm{sbp}(f(x)), \tag{5}$$

where $\mathrm{sbp}(\cdot): R \to R$ denotes a specially-constructed nonlinear mapping of neural dynamics and is defined as

$$\mathrm{spb}(y) = \mathrm{sgn}^r(y) + \mathrm{sgn}^{1/r}(y), \tag{6}$$

with parameter $r \in (0,1)$ and $\mathrm{sgn}^r(\cdot)$ defined as

$$\mathrm{sgn}^r(y) = \begin{cases} |y|^r & \text{if } y > 0; \\ 0 & \text{if } y = 0; \\ -|y|^r & \text{if } y < 0; \end{cases}$$

where $y \in R$ and $|y|$ denotes the absolute value of $y$.

### 2.3. Models comparison

While the above subsections present two different models for real-time nonlinear equation solving, in this subsection, the following details are given for comparing the OZD model (4) and the IZD model (5), as well as the GD model.

- A specially-constructed nonlinear function is exploited in the IZD model (5), which makes its form totally different from these of the GD model and the OZD model (4).
- The design of OZD model (4) and the IZD model (5) is based on the elimination of an indefinite error function. By contrast, the design of the GD model is based on the elimination of a square-based positive energy function.
- Similar to the OZD model (4), the IZD model (5) is depicted in implicit dynamics that is better consistent with the systems in practice and in nature. By contrast, the GD model is depicted in an explicit dynamics that is usually associated with classic Hopfield-type recurrent neural networks.
- It can be theoretically proved that the IZD model (5) achieves superior finite-time convergence performance in comparison with the GD model and the OZD model (4). In addition, for the case of a multiple root, with the increase of the order of the multiple root, the neural state of OZD model (4) and IZD model (5) can still converge well to such a theoretical root. In contrast, the GD model more probably yields wrong (or approximate) solutions [20,21].

Before ending this subsection, following the above third point, we would like to mention that the implicit dynamical systems frequently arise in analog electronic circuits and systems according to Kirchhoff's rules [20,21]. In addition, implicit dynamical systems have greater abilities in representing dynamical systems because of preserving physical parameters in the coefficient matrices, e.g., $f'(x)$ on the left-hand side of IZD model (5) and OZD model (4). Besides, the implicit systems could be mathematically transformed to explicit systems if needed. In this sense, owing to the advantages of implicit systems, our IZD model (5) and OZD model (4) can be much superior to the GD model in the form of explicit systems.

## 3. Convergence analysis

In this section, the finite-time convergent performance of IZD model (5) is investigated. Before that, the lemma [20,21] about global convergence of OZD model (4) for solving nonlinear equation is presented for comparative purpose.