Neurocomputing 149 (2015) 700–707

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Letters

Finite-time stability of fractional delayed neural networks

Ranchao Wu [a,*], Yanfen Lu [a], Liping Chen [b]

[a] School of Mathematics, Anhui University, Hefei 230601, China
[b] School of Electric Engineering and Automation, Hefei University of Technology, Hefei 230009, China

ARTICLE INFO

Article history:
Received 23 January 2014
Received in revised form
16 July 2014
Accepted 30 July 2014
Communicated by H. Zhang
Available online 11 August 2014

Keywords:
Neural networks
Fractional-order
Finite-time stability

ABSTRACT

In this paper, finite-time stability of a class of fractional delayed neural networks of retarded-type with commensurate order between 0 and 1 is investigated. For such problems in integer-order systems, Lyapunov functions are usually constructed, whereas no specific Lyapunov functions exist in fractional-order cases. By employing inequalities such as Hölder inequality, Gronwall inequalities and inequality scaling skills, some finite-time stability results are derived. For fractional delayed neural models of retarded-type with order $0 < \alpha < 0.5$ and $0.5 \le \alpha < 1$, sufficient conditions for the finite-time stability are presented. Numerical simulations also verify the theoretical results.

1. Introduction

Fractional calculus was firstly introduced 300 years ago, which mainly deals with derivatives and integrals of arbitrary order. Due to its complexity and lack of application background, it did not attract much attention for a long time. In recent decades, it has been proved to be valuable tools in modeling many phenomena in various fields of engineering, physics and economics [1–4]. It has been known that compared with classical integer-order models, fractional-order models could better describe the behavior of physical systems. For example, recently, fractional-order models of happiness [5] and love [6] have been developed and are claimed to give a better representation than the integer-order dynamical approaches. Since fractional derivatives are nonlocal and have weakly singular kernels [7], the major advantage is that they provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

Since fractional calculus has the memory, some researchers recently introduced it to neural networks to form fractional-order neural models, which could better describe the dynamical behavior of the neurons, such as “memory”. Also, it has been pointed out that fractional derivatives provide neurons with a fundamental and general computation ability that can contribute to efficient information processing, stimulus anticipation and frequency-independent phase shifts of oscillatory neuronal firing [8]. More recently, efforts have been made to investigate the complex dynamics of fractional-order neural networks. In [9], Arena et al. first introduced a cellular neural network with fractional-order cells. In [10], Petráš presented a fractional-order three-cell network which exhibits limit cycles and stable orbits for different parameter values. Furthermore, note that fractional-order recurrent neural networks might be expected to play an important role in parameter estimation [11–14]. Therefore, the incorporation of fractional derivatives and integrals into the neural networks is a great improvement in modeling and it is worthwhile to study.

As we know, stability analysis, such as Lyapunov stability, is one of central tasks in the study of fractional differential systems. Stability of neural networks has been widely investigated and excellent results have been obtained. In recent years, there have been some advances in stability theory of fractional differential systems [15–18]. On the other hand, since bifurcation and chaos of fractional-order neural networks were firstly investigated in [19,20], some important and interesting results about stability of fractional-order artificial neural networks have been obtained. For instance, in [21], the dynamics of fractional-order delay-free Hopfield neural networks, including stability and multi-stability, bifurcations and chaos, have been investigated. In [22], a fractional-order Hopfield neural model was proposed, and its stability was investigated by an energy-like function. Yu et al. [23] investigated α-stability and α-synchronization for fractional-order neural networks. Chaos and hyperchaos in fractional-order cellular neural networks were discussed in [24].

Nowadays much work has been done about the stability, such as Lyapunov stability, asymptotic stability, uniform stability, ultimately uniformly bounded stability and exponential stability, which are all concerned with the behavior of systems within an infinite time interval. However, in some practical applications the main concern is

* Corresponding author. Tel.: +86 551 63861484.
E-mail address: rcwu@ahu.edu.cn (R. Wu).

http://dx.doi.org/10.1016/j.neucom.2014.07.060
0925-2312/

the behavior of systems over a finite time interval. Actually most real neural systems only operate over finite time intervals. In such cases, it is necessary to care more about the finite-time behavior of systems than the asymptotic behavior over an infinite time interval. Therefore the finite-time stability was introduced, which was firstly put forward in the Russian literatures [25–27]. It could not only deal with systems with prescribed bounds and finite time intervals, but also obtain the estimated time of finite-time stability. All these are the advantages of finite-time stability over Lyapunov stability. During the 1960s, finite-time stability frequently appeared in the control literatures [28,29]. Until now, there are many valuable results about finite-time stability [30–36]. Recently, research about the finite-time behaviors could also be carried out on complex networks. But few results [37,38] are concerned with the finite-time stability of fractional-order neural networks. In [39], uniform stability of a class of fractional-order neural networks with delay was investigated. In [37], by the Laplace transform, the generalized Gronwall inequality and estimates of Mittag–Leffler functions, the finite-time stability of retarded-type cellular neural networks with order lying in (1,2) has been studied. However, for the models with order between 0 and 1, no specific Lyapunov functions exist for stability analysis; further the estimates in [37] could not be applied to such models. Motivated by the above discussions, to derive results about the finite-time stability, Hölder inequalities, Gronwall inequalities and inequality scaling skills are employed. Then sufficient conditions ensuring the finite-time stability of fractional retarded-type neural networks with order lying in (0,1) are derived.

The rest of the paper is organized as follows. Some necessary definitions and lemmas are recalled in Section 2. Two sufficient conditions ensuring the finite-time stability of fractional neural network with order $0<\alpha<0.5$ and $0.5\le\alpha<1$ in Section 3 are presented. Numerical simulations on a neural example are given in Section 4.

Throughout the paper, denote $\|x\|=\sum_{i=1}^{n}|x_i|$ and $\|A\|=\max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|$, which are the Euclidean vector norm and matrix norm, respectively; x_i and a_{ij} are the elements of the vector x and the matrix A, respectively. Denote the space of continuous functions mapping $[-\tau,0]$ into R^n by $\mathcal{C}$.

2. Preliminaries

In this section, some definitions, lemmas and well-known results about fractional calculus are recalled.

Definition 2.1 (Podlubny [1]). The fractional integral with non-integer order $\alpha>0$ of function $x(t)$ is defined as follows:

$$D_{t_0,t}^{-\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}x(\tau)\,d\tau,$$

where $\Gamma(\cdot)$ is the Gamma function $\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}\,dt$.

Definition 2.2 (Podlubny [1]). The Riemann–Liouville derivative of fractional order α of function $x(t)$ is given as

$${}_{RL}D_{t_0,t}^{\alpha}x(t)=\frac{d^k}{dt^k}D_{t_0,t}^{-(k-\alpha)}x(t)=\frac{d^k}{dt^k}\frac{1}{\Gamma(k-\alpha)}\int_{t_0}^{t}(t-\tau)^{k-\alpha-1}x(\tau)\,d\tau,$$

where $k-1<\alpha<k\in Z^+$.

Definition 2.3 (Podlubny [1]). The Caputo derivative of fractional order α of function $x(t)$ is defined as follows:

$${}_{C}D_{t_0,t}^{\alpha}x(t)=D_{t_0,t}^{-(k-\alpha)}\frac{d^k}{dt^k}x(t)=\frac{1}{\Gamma(k-\alpha)}\int_{t_0}^{t}(t-\tau)^{k-\alpha-1}x^{(k)}(\tau)\,d\tau,$$

where $k-1<\alpha<k\in Z^+$.

Lemma 2.1 (*Mitrinović and Dragoslav [40], Hölder inequality*). *Assume that $p,q>1$, and $1/p+1/q=1$, if $|f(\cdot)|^p$, $|g(\cdot)|^q\in L^1(E)$, then $f(\cdot)g(\cdot)\in L^1(E)$ and*

$$\int_E|f(x)g(x)|\,dx\le\left(\int_E|f(x)|^p\,dx\right)^{1/p}\left(\int_E|g(x)|^q\,dx\right)^{1/q}. \tag{1}$$

where $L^1(E)$ is the Banach space of all Lebesgue measurable functions $f:E\to R$ with $\int_E|f(x)|\,dx<\infty$.

Let $p,q=2$, it reduces to the Cauchy–Schwartz inequality as follows:

$$\left(\int_E|f(x)g(x)|\,dx\right)^2\le\int_E|f(x)|^2\,dx\int_E|g(x)|^2\,dx. \tag{2}$$

Lemma 2.2 (*Kuczma [41]*). *Let $n\in N$, and let $x_1,x_2,\ldots,x_n$ be non-negative real numbers. Then for $\omega>1$,*

$$\left(\sum_{i=1}^{n}x_i\right)^{\omega}\le n^{\omega-1}\sum_{i=1}^{n}x_i^{\omega}.$$

Lemma 2.3 (*Corduneanu [42], Gronwall inequality*). *If*

$$x(t)\le h(t)+\int_{t_0}^{t}k(s)x(s)\,ds,\quad t\in[t_0,T),$$

where all the functions involved are continuous on $[t_0,T)$, $T\le\infty$, and $k(t)\ge0$, then $x(t)$ satisfies

$$x(t)\le h(t)+\int_{t_0}^{t}k(s)h(s)\exp\left[\int_s^t k(u)\,du\right]ds,\quad t\in[t_0,T). \tag{3}$$

If, in addition, $h(t)$ is nondecreasing, then

$$x(t)\le h(t)\exp\left(\int_{t_0}^{t}k(s)\,ds\right),\quad t\in[t_0,T).$$

From the Laplace transform of fractional derivative, it is recognized that the main advantage of the Caputo derivative is that it only requires initial conditions given in terms of integer-order derivatives, representing well-understood features of physical situations and making it more applicable to real world problems. So in this paper, we deal with the fractional-order neural networks involving Caputo derivative, and the notation D^α is chosen as the Caputo fractional derivative operator ${}_CD_{0,t}^{\alpha}$.

The following properties of operator D^α are provided.

Lemma 2.4 (*Li and Deng [43]*). *If $x(t)\in\mathcal{C}^k[0,\infty)$, and $k-1<\alpha<k\in Z^+$, then*

(1) $D^{-\alpha}D^{-\beta}x(t)=D^{-(\alpha+\beta)}x(t),\quad \alpha,\beta\ge0,$
(2) $D^{\alpha}D^{-\alpha}x(t)=x(t),\quad \alpha\ge0,$
(3) $D^{-\alpha}D^{\alpha}x(t)=x(t)-\sum_{j=0}^{k-1}(t^j/i!)x^{(j)}(0),\quad \alpha\ge0.$

3. Main results

In this section, two sufficient conditions are derived for a class of fractional order neural networks with order $0<\alpha<0.5$ and $0.5\le\alpha<1$.

Consider the following fractional order cellular neural networks:

$$\begin{cases}D^{\alpha}x_i(t)=-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}(t)g_j(x_j(t-\tau))+I_i(t),\\ x_i(t)=\phi_i(t),\quad t\in[-\tau,0],\end{cases} \tag{4}$$