Neurocomputing 147 (2015) 280–290

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Stability and bifurcation analysis of reaction–diffusion neural networks with delays

Hongyong Zhao*, Jinglan Yuan, Xuebing Zhang

Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China

ARTICLE INFO

Article history:
Received 30 September 2013
Received in revised form
30 May 2014
Accepted 25 June 2014
Communicated by Z. Ozcan
Available online 19 July 2014

Keywords:
Reaction–diffusion
Neural networks
Stability
Delays
Bifurcation

ABSTRACT

In this paper, stability and Hopf bifurcation of reaction–diffusion neural networks with delays is considered, where the sum of the delays can be regarded as a bifurcation parameter. Some sufficient conditions are provided for checking stability and Hopf bifurcation. The particular attention is focused on the change of the stability as the bifurcation parameter τ increased. The computer simulations are provided to verify the efficiency of the theoretical results.

1. Introduction

Over the past decades, Hopfield neural networks [1,2] and their various generalizations [3–5] had attracted the attention of many researchers due to their potential applications in areas such as solving image processing, pattern recognition, associative memory, and optimization [6–9]. Since an exhaustive analysis of the dynamics of large systems is difficult, the smaller systems display similar behavior. Recently, Ruiz et al. [10] studied in detail the specific 3-node recurrent network described as

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1(t)+\tanh[x_2(t)] \\ \dfrac{dx_2}{dt} = -x_2(t)+\tanh[x_3(t)] \\ \dfrac{dx_3}{dt} = -x_3(t)+w_1\tanh[x_1(t)]+w_2\tanh[x_2(t)], \end{cases}$$

they had shown that the network model could possess an attractive limit cycle.

It is well known that there exits time delays in the information processing of neurons since the transmission of information from one neuron to another is not instantaneous. Time delays have important influences on the dynamical behavior of neural networks, they may destabilize the stable equilibria and admit periodic oscillation, bifurcation and chaos [11,12]. So in models of artificial neural networks, it is necessary to incorporate the processing time of each neuron to make them more realistic. Several papers were devoted to the stability and bifurcation of neural network models with delays [5,7,9,13]. Zou and his partners [14] studied the bifurcation of a three-unit neural network with delays

$$\begin{cases} \dfrac{dx_1}{dt} = -kx_1(t)+a\tanh[x_2(t-\tau_2)]+a\tanh[x_3(t-\tau_1)] \\ \dfrac{dx_2}{dt} = -kx_2(t)+a\tanh[x_3(t-\tau_2)]+a\tanh[x_1(t-\tau_1)] \\ \dfrac{dx_3}{dt} = -kx_3(t)+a\tanh[x_1(t-\tau_2)]+a\tanh[x_2(t-\tau_1)], \end{cases}$$

the authors gave the largest stability region of the trivial solution and the existence of codimension of one or two bifurcations of the system. Babcock and Westervelt [15] also suggested examining carefully the dynamical behavior of some simple networks. One of the simple networks they studied was the following two-neuron network model with two delays

$$\begin{cases} \dfrac{du_1}{dt} = -u_1(t)+a_1\tanh[u_2(t-\tau_2)] \\ \dfrac{du_2}{dt} = -u_2(t)+a_2\tanh[u_1(t-\tau_1)], \end{cases} \tag{1}$$

where τ_1 and τ_2 were delays, $\tanh(u)$ denoted the signal transmission function. Babcock and Westervelt showed that system (1) exhibited very interesting and rich dynamics including stable and unstable limit cycles, etc. Gopalsamy and Leung [16] also considered system (1) with $\tau_1=\tau_2$, the authors showed that under certain conditions, the delay induced a Hopf-type bifurcation.

* Corresponding author.
E-mail address: hongyongz@126.com (H. Zhao).

http://dx.doi.org/10.1016/j.neucom.2014.06.065
0925-2312/

The diffusion effects cannot be ignored in neural networks when electrons are moving in a nonuniform electromagnetic field. As pointed out the authors in [17–19], the whole structure and dynamic behaviors of multi-layer cellular neural networks are seriously dependent on the evolution time of each variable and its position (space), but also intensively dependent on its interactions deriving from the space-distributed structure of the whole networks. Therefore, the reaction–diffusion effects should be seriously considered. Recently, Liao et al. [20] discussed the stability of reaction–diffusion Hopfield neural networks without time delays. Wang and Xu [21,22] studied the global attractor of reaction–diffusion Hopfield neural networks with constant time delay and global exponential stability of reaction–diffusion Hopfield neural networks with time-varying delays, respectively. However, in all these works, the effect of diffusion on the stability or instability has not been considered. Moreover, only a few of works have discussed the bifurcation of two neurons model with diffusion terms and delays [23,24].

In designing a neural circuit, it is often desired that the neural network has a stable equilibrium point. As is well known, a neural network has complicated internal dynamics. It is therefore difficult to predict their behavior. For some special networks, when the parameters of the network have a particular symmetric structure and are chosen so that the overall dynamics of the network are asymptotically stable. However, if the parameters do not have a symmetric structure, then the analysis of the network dynamics becomes intractable [15,16]. It is therefore natural to ask how guarantee that the stability of equilibrium point? For asymmetric networks and symmetric networks, one of the purposes of the present paper is to give some parameter conditions which guarantee that the equilibrium point of the two-neuron reaction–diffusion model is asymptotically stable.

It is well known that studies on neural dynamical systems involve not only a discussion of stability properties, but also many dynamical behaviors such as periodicity and chaos. For example, in biological neural networks, the existence of periodic solutions causes some problems in neural network applications such as content addressable memory, periodic sequences of neural impulses are of fundamental importance for the control of motor body functions, such as heartbeat, which occurs with great regularity almost three billion times during an average person's life [15,16]. In artificial neural networks, the research of the periodicity may help to handle noises in hardware implementations [15,16]. Thus, seeking the conditions of periodicity is of both theoretical and practical importance for neural networks. The Hopf theory is one of the most important methods for studying periodic solutions in autonomous nonlinear neural systems. Therefore, another purpose of the present paper is to give some parameter conditions for which the two-neuron model with reaction–diffusion terms has a periodic solution based on ideas from bifurcation analysis. By such analytical forms, one may understand the given parameter conditions in the present paper mainly reflect the stability or periodicity of neural networks.

2. Neural networks' model for coupling connection terms with delays

Consider the following reaction–diffusion neural network model with delays

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = d_1\Delta u(t,x) - c_1 u(t,x) + a_1 g[v(t-\tau_1,x)] \\ \qquad + b_1 g[u(t,x)], \quad t>0, \quad x\in\Omega \\ \dfrac{\partial v(t,x)}{\partial t} = d_2\Delta v(t,x) - c_2 v(t,x) + a_2 g[u(t-\tau_2,x)] \\ \qquad + b_2 g[v(t,x)], \quad t>0, \quad x\in\Omega \\ \dfrac{\partial u(t,x)}{\partial n} = \dfrac{\partial v(t,x)}{\partial n} = 0, \quad t>0, \quad x\in\partial\Omega \\ u(t,x) = \Phi(t,x), v(t,x) = \Psi(t,x), \quad t\in[-\tau,0]\times\overline{\Omega}, \end{cases} \tag{2}$$

where $u(t,x)$ and $v(t,x)$ denote the activation state of the neuron; Ω is the one-dimensional spatial domain with $\Omega=(0,\pi)$, and $\partial/\partial n$ denotes the outward normal derivative; $\Delta: \partial^2/\partial x^2$ denotes the Laplacian operator; $d_i>0 (i=1,2)$ are the diffusion coefficients; $\tau_i (i=1,2)$ is the delay; $c_i>0 (i=1,2)$ denotes the self-feedback rate of the network; $a_i (i=1,2)$ and $b_i (i=1,2)$ denote the connection and self-feedback strength, respectively. $g(\cdot)$ is the activation function of neurons. $g(\cdot)$ is the C^1-smooth function with $g(0)=0$. $\Phi,\Psi\in C \triangleq C([-\tau,0],X)$ and X is defined by

$$X=\left\{u,v\in W^{2,2}(\Omega); \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0, x\in\partial\Omega\right\}$$

with $\langle\cdot,\cdot\rangle$ being the inner product. It is easy to see that system (2) always has a zero equilibrium point $(0,0)^T$. In the following, we analyze the stability and the bifurcation of the system (2) at $(0,0)^T$.

3. Asymptotical stability and Hopf bifurcation analysis for system (2)

Denote $U(t)=(u(t),v(t))^T=(u(t,\cdot),v(t,\cdot))^T$. Then system (2) can be rewritten as an abstract ordinary differential equation (ODE) in the Banach space C

$$\dot U(t) = D\Delta U(t) + L(U_t) + f(U_t), \tag{3}$$

where $D=diag\{d_1,d_2\}$, $L: C\to R^2$ and $f: C\times R^+\to R^2$.

The linear equation of system (2) at $(0,0)^T$ is

$$\dot U(t) = D\Delta U(t) + L(U_t), \tag{4}$$

whose characteristic equation is

$$\lambda y - D\Delta y - L(e^{\lambda\cdot}y)=0, \quad y\in dom(\Delta). \tag{5}$$

Obviously, the stability of zero equilibrium point of (2) depends on the locations of roots of characteristic equation (5). The zero equilibrium point of (2) is asymptotically stable if all roots of (5) have negative real parts. The zero equilibrium point of (2) is unstable if (5) has at least one root with a positive real part.

It is known that the eigenvalues of Δ on X are $-k^2 (k\in N_0\triangleq\{0,1,2,\ldots\})$ and the corresponding eigenfunctions are $\beta_k^1=(\gamma_k,0)^T$, $\beta_k^2=(0,\gamma_k)^T$, $\gamma_k=\cos(kx)$, and $\{\beta_k^1,\beta_k^2\}_{k=0}^\infty$ construct a basis of the phase space X, therefore, any element y in X can be expanded as a Fourier series in the following form:

$$y=\sum_{k=0}^\infty Y_k^T\begin{pmatrix}\beta_k^1\\ \beta_k^2\end{pmatrix}, \quad Y_k=\begin{pmatrix}\langle y,\beta_k^1\rangle\\ \langle y,\beta_k^2\rangle\end{pmatrix}. \tag{6}$$

In addition, some easy computations can show that

$$L(\varphi^T(\beta_k^1,\beta_k^2)^T) = L(\varphi)^T(\beta_k^1,\beta_k^2)^T, \quad k\in N_0, \tag{7}$$

for $\varphi=(\varphi_1,\varphi_1)^T\in C([-\tau,0],R^2)$.

For convenience, set $c_{11}=a_1g'(0)$, $c_{12}=b_1g'(0)$, $c_{21}=a_2g'(0)$, $c_{22}=b_2g'(0)$. From (6) and (7), (5) is equivalent to

$$\sum_{k=0}^\infty Y_k^T\left[(\lambda I_2+Dk^2)-\begin{pmatrix}-c_1+c_{12} & c_{21}e^{-\lambda\tau_2}\\ c_{11}e^{-\lambda\tau_1} & -c_2+c_{22}\end{pmatrix}\right]\begin{pmatrix}\beta_k^1\\ \beta_k^2\end{pmatrix}=0, \tag{8}$$

where $I_2=diag\{1,1,\ldots,1\}$.

Hence, (5) can be transformed into the following characteristic equation:

$$\lambda^2+(d_1k^2+c_1-c_{12}+d_2k^2+c_2-c_{22})\lambda \\ +(d_1k^2+c_1-c_{12})(d_2k^2+c_2-c_{22})-c_{11}c_{21}e^{-\lambda(\tau_1+\tau_2)}=0. \tag{9}$$

For simplicity, let $\tau=\tau_1+\tau_2$, $p_1(k^2)=(d_1+d_2)k^2+c_1-c_{12}+c_2-c_{22}$, $q_1(k^2)=(d_1k^2+c_1-c_{12})(d_2k^2+c_2-c_{22})$. Then the characteristic