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Stability and periodicity of discrete Hopfield neural networks with column arbitrary-magnitude-dominant weight matrix

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ABSTRACT

Stability and periodicity of neural networks is important behavior in biological and cognitive activities. In order to better simulate a biological genuine model, a special kind of discrete Hopfield neural networks (SDHNNs) in which every neuron has only one input is considered. By applying permutation theory and mathematical induction, we prove that the SDHNN always converges to a stable state or a limit cycle. The SDHNN is extended to the discrete Hopfield neural networks with column arbitrary-magnitude-dominant weight matrix (DHNNCAMDWM) in which there only exits a magnitude-dominant element in every column. Some important results, especially the periodic stability of the DHNNCAMDWM, are obtained. And the XOR problem is successfully solved by the results.

1. Introduction

Hopfield neural networks, as a signpost that is the second development of neural networks, have been extensively studied in past years because of their wide applicability in solving constraint satisfaction, optimization, associative memories, pattern classification, image processing, etc. These applications heavily depend on the dynamical behavior of the networks. So, the analysis of dynamical behaviors is very important in the design and application of Hopfield neural networks. Previous works mainly consider that the dominant effect of every neuron is the connected weight from itself to itself [11–16]. And in most references [1–16,36,37] the stability and periodicity of discrete Hopfield neural networks (DHNNs) is proved with the Lipschitz function. However, it is difficult to construct the Lipschitz function and the main influence of a neuron is not itself but others in most theories and practices. Therefore, in this paper we consider the stability and periodicity of SDHNN and DHNNCAMDWM using the permutation theory and mathematical induction (maybe it provides a new method for the provement of the stability and periodicity of DHNN).

With a common assumption that the weight matrix is symmetric, β-symmetric and anti-symmetric, scholars have done intensive studies about the Hopfield-type network. In most cases, the weight matrixes of neural networks are asymmetric, so to have strict restriction on symmetry is unpractical. In recent years, some studies about Hopfield-type network in which the weight matrix is asymmetric, mainly have two ways [1–21]. Firstly, some researchers attempt to look for some special asymmetric connected matrix, for examples: anti-symmetric matrix [1–3], semi-definiteness matrix [4], diagonal matrix [5–8], nonnegative matrix [9–16], and M-matrix [22,23]. Secondly, some researchers make efforts to analyze the degree distribution of the connection topology, for examples: poisson [17,18], small-world [19,20], scale-free [21]. We consider the former situation. The studies of the dominant weight matrix are as follow. Xu et al. [12–14] prove that any Hopfield networks will converge to a stable state when operating in the serial mode, if the connected matrix is main diagonally dominant matrix (MDDM); and the convergence of Hopfield networks in the parallel mode is also verified under corresponding nonnegative definiteness conditions. As the improvement of this study, Lee extends the serial (or fully parallel) updating mode to partial simultaneous updating mode [15]. And in the study of Ma et al., they add an increment matrix, row or column diagonally dominant, to the weight matrix [16]. Liang et al. [8] extend main diagonally dominant matrix to the

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quasi-diagonally row-sum dominant matrix (QDRDM) or quasi-diagonally column-sum dominant matrix (QDCDM).

From above we know that the dominant effect on every neuron is the connection weight from itself to itself. Applying MDDM, QDRDM and QDCDM, they mainly analyze the stability of Hopfield neural networks whose connected weight matrixes are main-diagonal dominant in these literatures [11–16]. However, the restriction is not suitable for the neuron on which the dominant effect is from any other. In this paper we consider DHNN, where each neuron only has one dominant-input neuron.

In addition, periodic oscillation in recurrent neural networks is an interesting behavior since many biological and cognitive activities require repetition [34]. Recently, many results have accumulated concerning the periodic oscillatory behavior of neural networks in the literature [24–34]. The periodic parameters of neural networks are required in the literature [24–32,34]. And there exists an important assumption that the weight matrix is a MDDM in [33]. However, the periodic behavior caused by the structure of DHNN has received very little research attention, despite of its practical importance. Simultaneously, scholars think that the feedback loops exist in networks and play a key role in network dynamics [17,18]. Therefore, it is important to study the periodicity of DHNNCAMDWM.

On other hand, in most references [1–16,36,37] stability analysis of these Hopfield neural networks mainly is based on the decrease of energy function, which typically requires that the weight matrix is nonnegative or negative definite. As you can s ee in example of Section 4, however, many column arbitrary-magnitude-dominant weight matrixs (CAMDWMs) are neither nonnegative definite nor negative definite. The stability of DHNNCAMDWM is obtained by researching a special kind of discrete Hopfield neural networks. Moreover, it is unnecessary to construct the Lipschitz function. So, in this paper the Lipschitz function method is more conservative.

Motivated by the previous discussion, the aims of this paper are to study the stability and periodicity of DHNNCAMDWM by permutation theory and mathematical induction. The method of this paper is as following. According to permutation theory, firstly, SDHNN P is divided into some smaller networks. Secondly, using all states of one neuron of SDHNN, which are divided into eight ways, the stability and periodicity of smaller networks is studied. Thirdly, based on those smaller networks the stability and periodicity of SDHNN is obtained by mathematical induction. Fourthly, this model is extended to DHNNCAMDWM. And the results are confirmed by solving the XOR problem.

This paper is organized as follows. In Section 2, we give some basic definitions. The stability and periodicity of the SDHNN are analyzed in Section 3. We study the stability of the DHNNCAMDWM in Section 4. An application is given in Section 5. The last section offers the conclusion of this paper.

2. Preliminaries

A discrete Hopfield neural networks $N=(W,\theta)$ consists of n binary valued nodes. We index the nodes by $H=\{1,2,\ldots,n\}$, and choose $\{1,-1\}$ as their possible states. Suppose that the state of neuron i at times t is $x_i(t)$. Then the state of n neurons can be represented by an n-dimensional vector $X(t)=(x_1(t),x_2(t),\ldots,x_n(t),)\in\{1,-1\}^n$. Let $x_j(t+1)$ be the state of neuron j at time $t+1$, then $x_j(t+1)$ and $x_i(t)(i=1,2,\ldots,n)$ has the relation

$$x_j(t+1)=\mathrm{sgn}\left(\sum_{i=1}^{n} w_{ij}x_i(t)-\theta_j\right)=\begin{cases}1, & \sum_{i=1}^{n} w_{ij}x_i(t)-\theta_j\geq 0\\ -1, & \sum_{i=1}^{n} w_{ij}x_i(t)-\theta_j<0\end{cases}\quad(1)$$

where $\theta=(\theta_1,\theta_2,\ldots,\theta_n,)^T\in R^n$ is an n-dimensional vector with θ_j representing the threshold value of neuron j and $W=(w_{ij})_{n\times n}$ is the connected matrix of the network with w_{ij} representing the linking strength (or weight) from neuron i to neuron j.

The network is updated asynchronously, that is, only one neuron i is selected at time $t+1$. The updating rule is

$$x_j(t+1)=\begin{cases}\mathrm{sgn}\left(\sum_{i=1}^{n} w_{ij}x_i(t)-\theta_j\right), & \text{if } j=i\\ x_j(t), & \text{if } j\neq i\end{cases}\quad(2)$$

The network is updated synchronously, that is, every neuron j is selected at time $t+1$. The updating rule is

$$x_j(t+1)=\mathrm{sgn}\left(\sum_{i=1}^{n} w_{ij}x_i(t)-\theta_j\right),\quad j=1,2,\ldots,n\quad(3)$$

In order to facilitate description, we denote that neuron set H is $\{1,2,\ldots,n\}$ and $f(j)$ is a mapping from H to itself. The following introduces the definition of SDHNN in which every neuron has only one input.

Definition 1. Let $P=\{W,\theta\}$ be a SDHNN, where W that every column has only one arbitrary number entry (others are zeros) is denoted by

$$w_{z_j j}=\begin{cases}\text{arbitrary number}, & \text{if } z_j=f(j)\\ 0, & \text{otherwise}\end{cases}\quad(4)$$

We know that a permutation of a set H is defined as a bijection from H to itself. According to the definition 1, f is not a bijection, so an extended permutation is defined as following:

Definition 2. An extended permutation d of a set H is defined as a mapping f from H to itself and denoted by

$$d=\begin{pmatrix}1 & 2 & \cdots & j & \cdots & n\\ z_1 & z_2 & \cdots & z_j & \cdots & z_n\end{pmatrix}\begin{matrix}1\leq j\leq n\\ z_j=f(j)\end{matrix}\quad(5)$$

And the corresponding cycles and directed links are written as

$$d=(H_1),\ldots,(H_b),\ldots,(H_L)=(H_{11}H_{12}),\ldots,(H_{b1}H_{b2}),\ldots,(H_{L1}H_{L2})\quad(6)$$

where $H_b=H_{b1}H_{b2}(1\leq b\leq L)$, H_{b1} consists of the neurons of a directed cycle and $H_{b2}(1\leq b\leq L)$ consists of the neurons of some directed links, which start from the neurons of H_{b1}.

Remark 1. When f is a bijection, d is a permutation. If we take out all links $H_{b2}(1\leq b\leq L)$, $d'=(H_{11}),\ldots,(H_{b1}),\ldots,(H_{L1})$ is a permutation. Understanding to Definition 1 and Definition 2, please see Example 1.

The corresponding entry $w_{z_j j}$ represents the connected weight from the neuron z_j to the neuron j. And the order n_b of $H_{b1}H_{b2}$ is the number of the neuron of the cycle H_{b1}.

In DHNN, each neuron only has one dominant-input neuron. That means the connected weight matrix of networks is a column arbitrary-dominant matrix in which every column only has one dominant element. Based on this, we define DHNN with column arbitrary-magnitude-dominant weight matrix (DHNNCAMDWM) as following:

Definition 3. Let $P=\{W,\theta\}$ be a DHNNCAMDWM, where W is a column arbitrary-magnitude-dominant weight matrix (CAMDWM). In every column of the matrix, the magnitude of any one entry in that column is larger than or equal to the sum of the magnitudes of all the other entries in that column. It is