Neurocomputing 122 (2013) 156–162

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Existence of solutions for fractional impulsive neutral functional differential equations with infinite delay ☆

Jiawu Liao [a], Fulai Chen [a,*], Sanqing Hu [b]

[a] Department of Mathematics, Xiangnan University, Chenzhou 423000, China
[b] College of Computer Science, Hangzhou Dianzi University, Hangzhou 310018, China

ARTICLE INFO

Article history:
Received 19 September 2012
Received in revised form
7 March 2013
Accepted 3 June 2013
Communicated by H. Zhang
Available online 12 July 2013

Keywords:
Fractional functional differential equation
Impulse
Existence of solutions
Hausdorff's measure of noncompactness

ABSTRACT

In this paper, we first introduce a class of impulsive neutral fractional functional differential equations which arise from many practical applications such as in viscoelasticity and electrochemistry. After providing a natural formula of solutions for the equations, we then give an existence theorem of the solutions by using the Hausdorff's measure of noncompactness and the theory of Mönch. As a result, the existence theory provides a theoretical basis for exploring the solutions of such kinds of differential equations.

1. Introduction

Fractional equations have received increasing attentions in recent years [1–13]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [1], and numerous applications of fractional differential equations can be found in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [13]. For example, in physics, the traditional way to deal with the behavior of certain materials under the influence of external forces in mechanics uses the laws of Hooke and Newton. If we are dealing with viscous liquids, then we can use Newton's law $\eta\varepsilon'(t)=\sigma(t)$, where $\sigma(t)$ and $\varepsilon(t)$ denote stress and strain at time t respectively, η is the so-called viscosity of the material. In view of all some possible interpolation properties, it is natural for us to design the classical Newton's law according to

$$\eta\,{}^{c}D_t^k\varepsilon(t)=\sigma(t),\quad k\in(n-1,n),\ \ n\in N,$$

which is called Nutting's law [14].

Impulsive effects exist widely in many processes in which their states are changed abruptly at certain time moment, these phenomena can be described in virtue of impulsive differential equations. A number of papers have been recently written on fractional order problems with impulsive effects [5,7,15–17]. Recently, Wang et al. [16] present a counterexample to show an error in the formula of solutions to a hybrid boundary value problem for an impulsive fractional differential equation in [5], the work is also supported by [18]. At the same time, the form of solutions should also be corrected for the initial value problem of impulsive fractional differential equation.

Motivated by the work in [16], in this paper, we give a new natural formula of solutions and establish the new existence theorems of the solutions for the initial value problem of the following fractional impulsive neutral functional differential equations with infinite delay:

$$\begin{cases} {}^{c}D_t^{\alpha}(x(t)+g(t,x_t))=f(t,x_t), & t\in J,\ t\neq t_k,\\ \Delta x(t_k)=I_k(x(t_k)), & t=t_k,\ k=1,2,\dots,m,\\ x(t)=\phi\in\mathcal{B}, & t\in(-\infty,0], \end{cases}\tag{1}$$

where ${}^{c}D_t^{\alpha}$ is the Caputo fractional derivative with $0<\alpha<1$, $\Delta x(t_k)=x(t_k^+)-x(t_k), 0=t_0<t_1<\cdots<t_m<t_{m+1}=b, J=[0,b]$, $f:J\times\mathcal{B}\to\mathbb{R}$ is Lebegue measurable with respect to t on J, $f(t,\phi), g(t,\phi)$ are continuous with respect to ϕ on $\mathcal{B}$, $I_k:\mathbb{R}\to\mathbb{R}$ are continuous for $k=1,\dots,m$, where $\mathcal{B}=\mathcal{B}((-\infty,0]\to\mathbb{R})$ denote the space of left piecewise continuous functions, and $x_t:(-\infty,0]\to\mathbb{R}$ is defined by $x_t(s)=x(t+s)$ for $-\infty<s\leq 0$, $\phi:(-\infty,0]\to\mathbb{R}$ belongs to $\mathcal{B}$.

☆This work was supported in part by the National Natural Science Foundation of Hunan Province under Grant 13JJ3120, and the Construct Program of the Key Discipline in Hunan Province, National Natural Science Foundation of China under Grant 61070127, and Qianjiang Project of Zhejiang Province, China, under Grant 2011R10063.

* Corresponding author.
E-mail addresses: ljwmath@yahoo.com.cn (J. Liao), cflmath@163.com (F. Chen), sqhu@hdu.edu.cn (S. Hu).

0925-2312/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.neucom.2013.06.034

This paper is organized as follows. In Section 2, we recall some useful preliminaries. In Section 3, we give a natural formula of solutions of (1) and discuss the existence of the solutions of (1) by the Hausdorff's measure of noncompactness and the theory of Mönch. We make conclusions in Section 4.

2. Preliminaries

In this paper, $\mathbb{R}$ is a Banach space, $L(\mathbb{R})$ denotes the Banach space of all linear and bounded functions on $\mathbb{R}$, and $C(J,\mathbb{R})$ is the space of all $\mathbb{R}$-valued continuous functions on J. Next we introduce several definitions and lemmas.

Definition 2.1 (*Hale and Kato [19]*). A linear space $\mathcal{B}$ consisting of functions from $(-\infty,0]$ into $\mathbb{R}$ with semi-norm $\|\cdot\|_{\mathcal{B}}$ is called an admissible phase space if $\mathcal{B}$ has the following properties:

(A_1) If $x(t):(-\infty,b]\to\mathbb{R}$ is continuous on J, and $x_0\in\mathcal{B}$, then $x_t\in\mathcal{B}$ and x_t is continuous on J, and

$$\|x(t)\|\le U\|x_t\|_{\mathcal{B}},$$

where $U\ge 0$ is a constant.

(A_2) There exist a continuous function $K(t)>0$ and a locally bounded function $M(t)\ge 0$ such that

$$\|x_t\|_{\mathcal{B}}\le K(t)\sup_{0\le s\le t}\|x(s)\|+M(t)\|x_0\|_{\mathcal{B}}$$

for $t\in[0,b]$ and $x(t)$ as in (A_1).

(A_3) $\mathcal{B}$ is a complete space.

Definition 2.2. The Hausdorff's measure of noncompactness χ_Y is denoted by $\chi_Y(\mathcal{B})=\inf\{r>0,\mathcal{B}$ can be covered by finite number of balls with radii $r\}$ for bounded set $\mathcal{B}$ in any Banach space Y.

In this paper we denote χ by the Hausdorff's measure of noncompactness of $\mathbb{R}$ and denote χ_C by the Hausdorff's measure of noncompactness of $C([0,b];\mathbb{R})$.

Lemma 2.1 (*Banas and Goebel [20]*). *If $W\subseteq C([0,b];\mathbb{R})$ is bounded, then*

$$\chi(W(t))\le\chi_C(W),\quad t\in[0,b]$$

where $W(t)=\{u(t),u\in W\}\subseteq\mathbb{R}$. Furthermore, if W is equicontinuous on $[0,b]$, then $\chi(W(t))$ is continuous on $[0,b]$ and

$$\chi_C(W)=\sup\{\chi(W(t)),t\in[0,b]\}.$$

To prove the existence of solutions of (1), we also need the following fixed point theorem: a nonlinear alternative of Mönch type.

Lemma 2.2 (*Mönch [21]*). *Let D be a closed convex subset of E, and let F be a continuous map from D into itself. If for some $x\in D$*

$$\overline{V}=\overline{co}(\{0\}\cup F(V))$$

implies that V is relatively compact for every countable subset V of D, then F has a fixed point.

To understand the theory of fractional differential equations, we also need the following definitions and lemmas.

Definition 2.3. The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I_t^{\gamma}f(t)=\frac{1}{\Gamma(\gamma)}\int_0^t\frac{f(s)}{(t-s)^{1-\gamma}}\,ds,\quad t>0,\gamma>0,$$

provided that the right-side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.4. The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f:[0,\infty)\to\mathbb{R}$ can be written as

$$D_t^{\gamma}f(t)=\frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{\gamma+1-n}}\,ds.$$

Definition 2.5. Caputo's derivative of order γ for a function $f:[0,\infty)\to\mathbb{R}$ can be written as

$${}^{c}D_{\gamma}^{t}f(t)=D_t^{\gamma}f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\Gamma(k-\gamma+1)}t^{k-\gamma},$$

$$t>0,\quad n-1<\gamma<n.$$

Lemma 2.3 (*Zhang [22]*). *Let $n-1<\gamma<n$, then the differential equation*

$${}^{c}D_{\gamma}^{t}x(t)=0$$

has solutions $x(t)=c_0+c_1t+c_2t^2+\cdots+c_nt^{n-1},c_i\in\mathbb{R},i=0,1,\cdots,n.$

Lemma 2.4 (*Zhang [22]*). *Let $n-1<\gamma<n$, then*

$$I_t^{\gamma\,c}D_{\gamma}^{t}x(t)=x(t)+c_0+c_1t+c_2t^2+\cdots+c_nt^{n-1}$$

for some $c_i\in\mathbb{R},i=0,1,\ldots,n.$

Lemma 2.5 (*Chen [17]*). *Let $0<\alpha<1$ and let $h:[0,b]\to\mathbb{R}$ is continuous, a function $x\in C([0,b],\mathbb{R})$ is a solution of the fractional Cauchy problems*

$$\begin{cases}{}^{c}D_t^{\alpha}x(t)=h(t), & t\in[0,b],\\ x(t_0)=x_0, & 0<t_0<b,\end{cases}\tag{2}$$

if and only if x is a solution of the following fractional integral equation:

$$x(t)=x(t_0)-\frac{1}{\Gamma(\alpha)}\int_0^{t_0}(t_0-s)^{\alpha-1}h(s)\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)\,ds.\tag{3}$$

3. Main result

Let $\tilde{\mathcal{B}}$ be set defined by

$$\tilde{\mathcal{B}}=\{x:(-\infty,b]\to\mathbb{R},x_k\in C(J_k,\mathbb{R}),\quad k=0,1,\ldots,m\},$$

where x_k is defined on $J_0=[0,t_1],J_k=(t_k,t_{k+1}],k=1,2,\ldots,m$. Set $\|\cdot\|_0$ be a semi-norm in $\tilde{\mathcal{B}}$ defined by

$$\|x\|_0=\|\phi\|_{\mathcal{B}}+\sup\{\|x(t)\|:t\ge 0\},\quad x\in\tilde{\mathcal{B}}\quad\text{and}\quad\phi\in\mathcal{B}.$$

Lemma 3.1. *A function $x\in\tilde{\mathcal{B}}$ is a solution of (1), if and only if $x\in\tilde{\mathcal{B}}$ is a solution of the following fractional integral equations:*

$$x(t)=\begin{cases}\phi, & t\in(-\infty,0],\\ \phi(0)+g(0,\phi(0))-g(t,x_t)\\ \quad+\dfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_s)\,ds, & t\in[0,t_1],\\ \phi(0)+g(0,\phi(0))-g(t,x_t)\\ \quad+\dfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_s)\,ds+\sum_{i=1}^{k}I_i(x(t_i)), & t\in(t_k,t_{k+1}].\end{cases}\tag{4}$$

Proof. Suppose that x is a solution of (1).

For $t\in[0,t_1]$, applying Lemma 2.4, we have

$$x(t)=\phi(0)+g(0,\phi(0))-g(t,x_t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_s)\,ds,$$