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# Multiple bifurcations in a harmonic oscillator with delayed feedback ☆

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ABSTRACT

In this paper, we consider a harmonic oscillator with delayed feedback. By studying the distribution of the eigenvalues of the characteristic equation, we drive the critical values where Bogdanov–Takens (B–T) bifurcation and zero-Hopf bifurcation occur. The versal unfoldings of the normal forms at the singularity of B–T and a pure imaginary and a zero eigenvalue singularity are given, respectively. Some numerical simulations verify the theoretical results.



## 1. Introduction

As is known to all, second order delay differential equations arise in a variety of mechanical, circuit system in which inertia plays an important role, such as harmonic oscillator, van der Pol' oscillator, etc., and many of them are regulated by feedback which depends on the state and the derivative of the state. Recently, the interest in studying of nonlinear dynamical system with delay has been growing rapidly (see [1–3,5–7,12–19] and the references therein). For example, in [3,12,14], the authors investigate pitchfork-Hopf bifurcation and B–T bifurcation for van der Pol's oscillator with delayed feedback, and obtain the complete bifurcation diagram for original parameter of the system by using the normal form method. Song et al. [16] study the following damped harmonic oscillator with delayed feedback:

$$\ddot{x}(t)+b\dot{x}(t)+ax(t)=f[x(t-\tau)],$$

and demonstrate steady-state bifurcation, B–T bifurcation, triple zero and Hopf-zero singularities by taking the time delay as the bifurcation parameter. In [19], taking the delay as a bifurcation parameter, the authors study the Hopf bifurcation of the following system:

$$\ddot{x}(t)+c\dot{x}(t)+kx(t)=s_1f(x(t-\tau))+s_2f(\dot{x}(t-\tau)), \tag{1}$$

where $c\ge 0, k>0$ are damping and stiffness of the system, respectively; $\tau$ is the time delay; $s_1, s_2$ are feedback gains, $f$ is a nonlinear force input function; and Campbell et al. [6,7] investigate the Hopf and resonant codimension two bifurcation. However, there are few papers discussing B–T and the zero-Hopf bifurcation of the harmonic oscillator model (1). This fact motivates our work for the paper. On the other hand, zero-Hopf and B–T singularity analysis on a system is a useful approach that can provide much information about dynamical behavior.

For notational convenience, let $x_1(t)=x(t), x_2(t)=\dot{x}(t)$, we can rewrite Eq. (1) in the following form:

$$\begin{cases} \dot{x_1}(t)=x_2(t) \\ \dot{x_2}(t)=-kx_1(t)-cx_2(t)+s_1f(x_1(t-\tau))+s_2f(x_2(t-\tau)). \end{cases} \tag{2}$$

In the present paper, we always assume that the function $f$ satisfies the following conditions:

**(H1).** $f\in C^3(\mathbb{R}),\ f(0)=0.$

The objective of this manuscript is to study the bifurcation of system (1) for B–T and zero-Hopf bifurcation. In Section 2, we analyze the distribution of the eigenvalues of corresponding transcendental characteristic equation of its linearized equation, and obtain the critical values for B–T and zero-Hopf bifurcation. In Section 3, we perform the center manifold reduction and normal form computation, and derive the normal forms with the B–T and zero-Hopf singularity for the harmonic oscillator. In Section 4, some examples are given and numerical simulations are performed to illustrate the obtained results. In Section 5, we summarize our results.

## 2. The analysis of eigenvalues

Clearly, $(0,0)$ is always the equilibrium of Eq. (2). Linearizing Eq. (2) at the origin yields the following system:

$$\begin{cases} \dot{x_1}(t)=x_2(t) \\ \dot{x_2}(t)=-kx_1(t)-cx_2(t)+s_1l_1x_1(t-\tau)+s_2l_1x_2(t-\tau), \end{cases} \tag{3}$$

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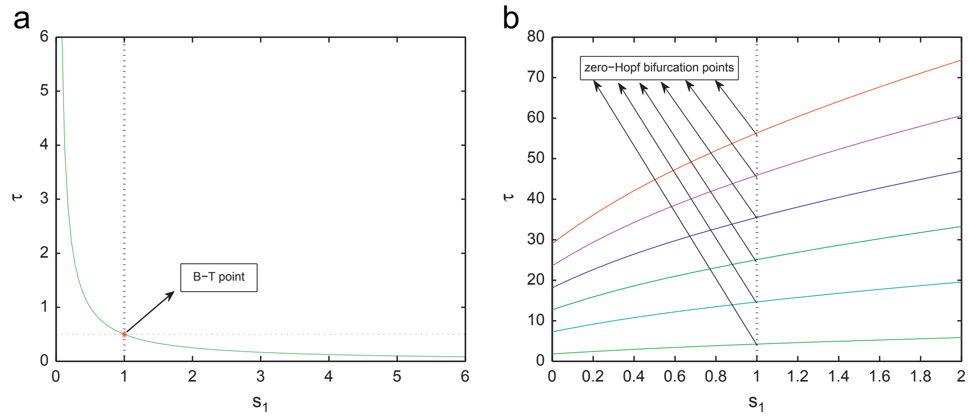


**Fig. 1.** The bifurcation diagram in the $s_1 - \tau$ plane.

where $l_1 = f'(0)$. Characteristic equation for Eq. (3) is

$$\det\begin{bmatrix} \lambda & -1 \\ k - s_1 l_1 e^{-\lambda\tau} & \lambda + c - s_2 l_1 e^{-\lambda\tau} \end{bmatrix} = 0.$$

Hence, the following second order exponential polynomial equation is obtained:

$$\Delta(\lambda,\tau) = \lambda^2 + c\lambda + k - (s_1 l_1 + s_2 l_1 \lambda) e^{-\lambda\tau} = 0. \tag{4}$$

To establish our main results, it is necessary to make the following assumptions:

**(H2)** $s_1 l_1 = k$.

Then, we have the following lemma.

**Lemma 1.** *Suppose that* (H2) *is satisfied, then*

(i) *if* $\tau \neq \tau^* := (s_2 l_1 - c)/k$, $\lambda = 0$ *is a single root of Eq.* (4);

(ii) *if* $\tau = \tau^*$ *and* $\tau^* \neq (s_2 l_1 \pm \sqrt{s_2^2 l_1^2 + 2k})/k$, $\lambda = 0$ *is a double root to Eq.* (4).

**Proof.** Clearly, $\lambda = 0$ is a root to Eq. (4) if and only if (H2) is satisfied. Substituting $s_1 l_1 = k$ into $\Delta(\lambda,\tau)$ and taking the derivative with respect to $\lambda$ give

$$\frac{d\Delta(0,\tau)}{d\lambda} = c + k\tau - s_2 l_1.$$

For any $\tau > 0$, if $\tau \neq \tau^*$, it is easy to see that $d\Delta(0,\tau)/d\lambda \neq 0$, which implies the conclusion of (i) follows. Furthermore, we have

$$\frac{d^2\Delta(0,\tau)}{d\lambda^2} = -k\tau^2 + 2 s_2 l_1 \tau + 2,$$

clearly, if $\tau^* \neq (s_2 l_1 \pm \sqrt{s_2^2 l_1^2 + 2k})/s_1 l_1$, then $d^2\Delta(0,\tau)/d\lambda^2 \neq 0$. This completes the proof. □

In the following, we consider the case that Eq. (4) not only has a zero root, but also has a pair of purely imaginary root $\pm i\omega$. Then we make the following assumption:

**(H3)** $s_2^2 l_1^2 + 2k - c^2 > 0$.

Based on (H2), let $i\omega (\omega > 0)$ be a root of Eq. (4) and separating the real and imaginary parts, we have that

$$\begin{cases} -\omega^2 + k - k\cos\omega\tau - s_2 l_1 \omega \sin\omega\tau = 0, \\ c\omega + k\sin\omega\tau - s_2 l_1 \omega \cos\omega\tau = 0, \end{cases} \tag{5}$$

adding squares of two equations yields

$$\omega^2(\omega^2 + (c^2 - 2k - s_2^2 l_1^2)) = 0. \tag{6}$$

Clearly, Eq. (6) has a positive root if and only if (H3) is satisfied. Furthermore, we can solve $\omega$ from above equation

$$\omega = \omega_0 = \sqrt{s_2^2 l_1^2 + 2k - c^2},$$

and from Eq. (5), we can get

$$\cos(\omega_0\tau) = \frac{-k\omega_0^2 + c s_2 l_1 \omega_0^2 + k^2}{k^2 + s_2^2 l_1^2 \omega_0^2},$$

then

$$\tau = \tau_j = \frac{1}{\omega_0}\left[\arccos\left(\frac{-k\omega_0^2 + c\omega_0^2 s_2 l_1 + k^2}{k^2 + s_2^2 l_1^2 \omega_0^2}\right) + 2j\pi\right].$$

**Lemma 2** (*see Ruan and Wei [9]*). *Consider the exponential polynomial*

$$\begin{aligned} P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}) = {} & \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda \\ & + p_n^{(0)} + [p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}] e^{-\lambda\tau_1} \\ & + \cdots + [p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}] e^{-\lambda\tau_m}, \end{aligned}$$

*where* $\tau_i \geq 0 (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots n)$ *are constants. As* $(\tau_1, \tau_2, \cdots, \tau_m)$ *vary, the sum of the order of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ *on the open right half plane can change only if a zero appears on or crosses the imaginary axis.*

In order to obtain our main result, we make the following assumption:

**(H4).** $s_2 l_1 < c$.

**Lemma 3.** *All the roots of Eq.* (4), *except the zero root, have negative real parts when* (H2) *and* (H4) *satisfied, and* $0 < \tau < \tau_0$.

**Proof.** For $\tau = 0$, Eq. (4) can be transformed into the following form:

$$\lambda^2 + (c - s_2 l_1)\lambda + k - s_1 l_1 = 0. \tag{7}$$

Since (H2) and (H4) hold, we get that the roots of Eq. (7) are $\lambda_1 = 0$, $\lambda_2 = s_2 l_1 - c < 0$. Using Lemma 2, we complete the proof. □

**Remark 1.** If (H4) satisfied, we can easily obtain $\tau^* = (s_2 l_1 - c)/k < 0$. Since $\tau \geq 0$, then $\lambda = 0$ is always a single root to Eq. (4). Therefore, B–T bifurcation and zero-Hopf bifurcation cannot coexist.

Summarizing the discussions above, we have the following theorem.