Neurocomputing 74 (2011) 949–953

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Stability analysis of stochastic recurrent neural networks with unbounded time-varying delays ☆

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ARTICLE INFO

Article history:
Received 21 June 2010
Received in revised form
11 October 2010
Accepted 6 November 2010
Communicated by J. Liang
Available online 23 December 2010

Keywords:
Stochastic recurrent neural networks
Unbounded delay
Moment stability
Almost sure stability
M-matrix

ABSTRACT

In this paper, the stability analysis issue of stochastic recurrent neural networks with unbounded time-varying delays is investigated. By the idea of Lyapunov function and the semi-martingale convergence theorem, both pth moment exponential stability and almost sure exponential stability are obtained. Moreover, the M-matrix technique is borrowed to make the results more applicable. Our criteria can be used not only in the case of bounded delay but also in the case of unbounded delay. Some earlier results are improved and generalized. An example is also given to demonstrate our results.

1. Introduction

To date, delayed recurrent neural networks have been extensively studied in the literature, see, for example, Refs. [1,4,10,11,13,16], where the time delays under consideration can be classified as constant delays, time-varying delays and distributed delays. Usually, constant fixed time delays in the models of delayed feedback systems serve as good approximation in simple circuits having a small number of cells [24]. In delayed neural networks, a constant delay is only a special case. In most situations, delays are variable, and in fact unbounded. That is, the entire history affects the present. In some practical applications and hardware implementations of neural networks, the inevitable time delay may be unbounded. Therefore, the studies of neural networks with time-varying delays and unbounded time delays are more important and necessary than those with constant delays, and the corresponding research can be seen in [21,23].

In fact, a real system is usually affected by external perturbations which have great uncertainty and hence may be treated randomly [22], as pointed out by Haykin [6] that in real neural systems, the synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. Hence, it is important to consider stochastic effects to the dynamical behaviors of neural networks. Liao and Mao [14,15] initiated the study of stability and instability of stochastic neural networks. In recent years, many researchers have a lot of contributions to the stability analysis issue for stochastic recurrent neural networks (SRNNs) with time delays. However, to the best of our knowledge, there are no corresponding results of SRNNs with unbounded time-varying delays.

Motivated by the above discussions, in this paper we consider the following SRNNs model:

$$dx_i(t)=\left[-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}g_j(t,x_j(t-\delta_j(t)))\right]dt+\sum_{j=1}^{n}\sigma_{ij}(t,x_j(t),x_j(t-\delta_j(t)))dw_j(t),\quad i=1,2,\ldots,n. \tag{1.1}$$

where $\delta_j(t)(j=1,\ldots,n)$ are delay functions which may be unbounded. Denote $y(t)=(y_1(t),y_2(t),\ldots,y_n(t))^{\mathrm{T}}=(x_1(t-\delta_1(t)),x_2(t-\delta_2(t)),\ldots,x_n(t-\delta_n(t)))^{\mathrm{T}}$. Eq. (1.1) can be rewritten as

$$dx(t)=[-Cx(t)+Af(x(t))+Bg(t,y(t))]dt+\sigma(t,x(t),y(t))\,dw(t), \tag{1.2}$$

where $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^{\mathrm{T}}\in\mathbb{R}^n$ is the state vector associated with the neurons; $C=\mathrm{diag}(c_1,c_2,\ldots,c_n)$ with $c_i>0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and the external

☆This research was supported by the fundamental research funds for the central universities under grant No.2010MS130.

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doi:10.1016/j.neucom.2010.11.022

stochastic perturbations; $A=(a_{ij})_{n\times n}$ and $B=(b_{ij})_{n\times n}$ represent the connection weight matrix and the delayed connection weight matrix, respectively; f_i and g_i are activation functions, $f(x(t))=(f_1(x_1(t)), f_2(x_2(t)),\ldots,f_n(x_n(t)))^T\in\mathbb{R}^n$, $g(t,y(t))=(g_1(t,x_1(t-\delta_1(t))), g_2(t,x_2(t-\delta_2(t))),\ldots,g_n(t,x_n(t-\delta_n(t))))^T\in\mathbb{R}^n$. Let $\tau=\max_{1\le i\le n}\delta_i(0)$. $\sigma=(\sigma_{ij})_{n\times n}$ is the diffusion coefficient matrix.

In [19], Sun and Cao investigated Eq. (1.2) with bounded delay functions $\delta_j(t)(j=1,\ldots,n)$. Here, we extend the delay functions $\delta_j(t)$ to unbounded delays and give the criteria on both pth moment exponential stability and almost sure exponential stability. The main results can be described by the following inequalities:

$$\limsup_{t\to\infty}\frac{\log\mathbb{E}|x(t,\xi)|^p}{t}\le -q, \tag{1.3}$$

$$\limsup_{t\to\infty}\frac{\log|x(t,\xi)|}{t}\le -\frac{q}{p}, \quad a.s., \tag{1.4}$$

where p, $q>0$ are independent of the initial data ξ.

Many methods have been exploited to study the stability in the publications. Such as the method of variation parameter [19], Halanay-type inequality [8], the linear matrix inequality (LMI) approach [18], and Razumikhin method [26]. However, these techniques cannot extend to the case of unbounded delays directly. To overcome difficulties from unbounded delays, we develop several new techniques. By virtue of the M-matrix, several useful criteria are obtained. These criteria are described only in terms of given system parameters and hence are extremely useful in applications.

In the next section, we give some preliminaries. The main results of this paper are developed in Section 3 where several sufficient criteria are established for moment exponential stability and almost sure exponential stability. Finally, we consider an example to illustrate our results. The example shows that our criteria can be used not only in the case of bounded delay but also in the case of unbounded delay.

2. Preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, denote its trace norm by $|A|=\sqrt{\text{trace}(A^TA)}$. For any given $x\in\mathbb{R}^n$, assume $x=(x_1,x_2,\ldots,x_n)^T$, $x\gg 0\Leftrightarrow x_i>0$ $(1\le i\le n)$. Let $\mathbb{R}_+=[0,\infty)$ and $\mathbb{R}^n_{++}=\{x\in\mathbb{R}^n: x\gg 0\}$. For any $c=(c_1,\cdots,c_n)^T\in\mathbb{R}^n$, let $\text{diag}(c)=\text{diag}(c_i)=\text{diag}(c_1,c_2,\cdots,c_n)$ denote the $n\times n$ matrix with all elements zero except those on the diagonal which are $c_1,\ldots,c_n$. In this paper, *const* always represents some positive constant whose value is not important.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets. Let $w(t)$ be an n-dimensional Brownian motion defined on this probability space.

Let $C^2(\mathbb{R}^n;\mathbb{R}_+)$ denote the family of all functions from $\mathbb{R}^n$ to $\mathbb{R}_+$ which are continuously twice differentiable. For any $V(x)\in C^2(\mathbb{R}^n;\mathbb{R}_+)$, we define a function $\mathcal{L}V:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ by

$$\mathcal{L}V(t,x,y)=V_x(x)[-Cx+Af(x)+Bg(t,y)]+\tfrac{1}{2}\text{trace}[\sigma^T(t,x,y)V_{xx}(x)\sigma(t,x,y)], \tag{2.1}$$

where

$$V_x(x)=\left(\frac{\partial V(x)}{\partial x_1},\frac{\partial V(x)}{\partial x_2},\cdots,\frac{\partial V(x)}{\partial x_n}\right), V_{xx}(x)=\left[\frac{\partial^2 V(x)}{\partial x_i\partial x_j}\right]_{n\times n}.$$

If $x(t)$ is a solution of Eq. (1.2), by the Itô formula,

$$dV(x(t))=LV(x(t))\,dt+V_x(x(t))\sigma(t,x(t),y(t))\,dw(t),$$

where $LV(x(t))=\mathcal{L}V(t,x(t),y(t))$.

In this paper, denote $\Delta_i(t)=t-\delta_i(t)$. We assume that $\delta_i(t)\in C^1(\mathbb{R}_+;\mathbb{R}_+)$, and

$$\eta_i:=\inf_{t\ge 0}\Delta'_i(t)>0, \quad (1\le i\le n) \tag{2.2}$$

which clearly shows that $\Delta_i(t)$ is a strictly increasing function on $[0,\infty)$ and has the inverse function $\Delta_i^{-1}(s)$ defined on $[-\delta_i(0),\infty)$ with the following property:

$$[\Delta_i^{-1}(s)]'=\frac{1}{\Delta'_i(t)}\le\eta_i^{-1}. \quad (s=\Delta_i(t)) \tag{2.3}$$

Lemma 2.1. *Let η_i be defined in (2.2), then $\eta_i\le 1(i=1,\ldots,n)$.*

Proof. By (2.2), noting that $\delta_i(t)\ge 0$, we have $\Delta'_i(t)\ge\eta_i$ and $\Delta_i(t)\le t$ for all $t\ge 0$. Then for any $t>0$

$$\eta_i\le\frac{1}{t}\int_0^t\Delta'_i(s)\,ds=\frac{\Delta_i(t)-\Delta_i(0)}{t}\le\frac{t+\tau}{t},$$

where $\tau=\max_{1\le i\le n}\delta_i(0)$. Letting $t\to\infty$, we have $\eta_i\le 1$. □

Denote by $C=C([-\tau,0],\mathbb{R}^n)$ the family of all continuous functions from $[-\tau,0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\|=\sup_{-\tau\le\theta\le 0}|\varphi(\theta)|$, which forms a Banach space. If $\tau=0$, then $C=\mathbb{R}^n$.

Throughout this paper, the following standard hypothesis are needed:

(H1) $f_i(u)$ satisfies the Lipschitz condition, that is, for each $i=1,2,\ldots,n$, there exists a constant $s_i>0$ such that
$$|f_i(u)-f_i(v)|\le s_i|x-y|, \quad \forall u,v\in\mathbb{R}.$$

(H2) For each $i=1,2,\ldots,n$, there exist constants $\varepsilon,r_i>0$ such that
$$|g_i(t,u)|\le r_i|u|e^{-\varepsilon\delta_i(t)}, \quad t\ge 0,u\in\mathbb{R}.$$

(H3) For each $i,j=1,2,\ldots,n$, there exist constants $\varepsilon,\lambda_{ij},\overline{\lambda}_{ij}\ge 0$ such that
$$\sigma_{ij}^2(t,u,v)\le\lambda_{ij}u^2+\overline{\lambda}_{ij}v^2e^{-\varepsilon\delta_j(t)}, t\ge 0,u,v\in\mathbb{R}.$$

Remark 2.2. Since $e^{-\varepsilon\delta_i(t)}$ is decreasing in ε, the above conditions will still hold when ε is replaced by any $\varepsilon'\in(0,\varepsilon]$. If $\tau_i=\sup_{t\ge 0}\delta_i(t)<\infty$ $(i=1,2,\ldots,n)$, then $e^{-\varepsilon\delta_i(t)}\ge e^{-\varepsilon\tau_i}\to 1$ as $\varepsilon\to 0$. Hence, when $\tau_i<\infty$, conditions (H2) and (H3) do not need the terms $e^{-\varepsilon\delta_i(t)}$ $(i=1,\ldots,n)$, namely, terms $e^{-\varepsilon\delta_i(t)}$ play a role only when $\tau_i=\infty$.

For the purpose of stability, let $f(0)\equiv 0, g(t,0)\equiv 0, \sigma(t,0,0)\equiv 0$, which shows that Eq. (1.2) admits a trivial solution. In this paper, we also assume that g and σ satisfy the local Lipschitz condition.

Let $Q=[q_{ij}]\in\mathbb{R}^{n\times n}$, $q_{ij}\le 0<q_{ii}$ for $i,j=1,2,\ldots,n, i\ne j$. Q is called an M-matrix if all the eigenvalues of Q have positive real parts. There are many conditions which are equivalent to the statement that Q is an M-matrix and we now cite some of them for the use of this paper. For more detailed information, please see [3].

Lemma 2.3. *Assume that $Q=[q_{ij}]\in\mathbb{R}^{n\times n}$, $q_{ij}\le 0<q_{ii}$ $(i\ne j, i,j=1,\ldots,n)$, then the following statements are equivalent:*

(i) *Q is an M-matrix.*
(ii) *There exists $c\in\mathbb{R}^n_{++}$ such that $Q^Tc\in\mathbb{R}^n_{++}$.*
(iii) *All the leading principal minors of Q are positive.*

Then we give the continuous semi-martingale convergence theorem (cf. [17]).