Neurocomputing 168 (2015) 475–487

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Hopf bifurcation and spatio-temporal patterns in a hierarchical network with delays and $\mathbb{Z}_2 \times \mathbb{Z}_n$ symmetry

Haijun Hu [a], Yanxiang Tan [a,b], Chuangxia Huang [a,c,*]

[a] School of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha, Hunan 410114, PR China
[b] College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China
[c] Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing 10090, China

ARTICLE INFO

Article history:
Received 13 December 2014
Received in revised form
26 March 2015
Accepted 20 May 2015
Communicated by Yang Tang
Available online 30 May 2015

Keywords:
Hierarchical network
Hopf bifurcation
Normal form
Symmetry
Time delay

ABSTRACT

A hierarchical network composed of two interacting rings each of which consists of n identical cells with an unidirectional coupling is the topic of this paper. We present a detailed discussion about the linear stability of the equilibrium by analyzing the associated characteristic equation. The local Hopf bifurcation and spatio-temporal patterns of bifurcating periodic oscillations are also given by employing the symmetric Hopf bifurcation theory for delay differential equations. In particular, by using the normal form theory and the center manifold theorem, we derive the formula determining the direction of the Hopf bifurcation and the stability of the bifurcated periodic orbits. An example with numerical simulations is presented to illustrate our theoretical results.

1. Introduction

There has been a flurry of research activity on neural networks since Hopfield [1] constructed a simplified neural network model. In this model, each neuron is represented by a linear circuit consisting of a resistor and a capacitor and is connected to the other neurons via sigmoidal activation functions. This model can be regarded as an information processing device that is inspired by the way biological nervous systems, such as the brain, process information simultaneously. However, the information transmission between neurons is not instantaneous and so it is more reasonable and realistic for the model to incorporate the factor of time delays (see [2]). A variety of research carried on neural network models with delays has shown that delays can lead to interesting dynamics in various ways. For more details, we refer to [3–12], and references therein. In addition, due to the complexity of dynamics of neural networks, some recent works (e.g., [13–18]) have focused on networks with the same time delay, the small scale or simple architectures.

Ring networks have been found in a variety of neural architectures such as cerebellum [19], and even in the fields of chemistry and electrical engineering. In fact, ring networks are of a limited biological relevance, and may be regarded as building blocks for networks with more realistic connection topologies. Among many models of neural network, ring networks can lead to many interesting patterns of oscillation. Thus, they can be studied to gain some insight into the mechanisms underlying the behavior of recurrent network [2,20]. In recent years, a ring structure with nearest-neighbor (unidirectional or bidirectional) coupling between the elements has received a great deal of attention; a significant body of research has been carried out (see [15–17,21–25] and references therein). Some of these studies have concerned lower dimensional systems. For example, Guo [25] studied a tri-neuron ring model with unidirectional coupling

$$\dot{u}_i(t) = -\mu u_i(t) + f(u_{i+1}(t-\tau)), \quad i (\mathrm{mod}\ 3), \tag{1.1}$$

where $\dot{u} = du/dt$, $u_i(t)$ represents the activation of the ith neuron at time t, $\mu > 0$ represents the decay rate of the activation, f represents the activation function, $\tau \geq 0$ is the signal transmission delay.

The vast majority of previous works have just considered the individual network but not investigated the interactions between multiple networks. In fact, numerous natural and artificial systems possess a hierarchic structure or functioning and can be naturally described by coupled sub-network. Coupled networks of nonlinear dynamical systems can exhibit rich dynamics, such as synchronization, symmetric bifurcation, chaos (see [26–31]). The rich dynamics arising from the interaction of simple networks can help scientists analyze the collective behavior of complex systems. For example, the brain may be conceived as a dynamic network of

* Corresponding author at: School of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha, Hunan 410114, PR China.
E-mail addresses: huhaijun2000@163.com (H. Hu), cxiahuang@126.com (C. Huang).

http://dx.doi.org/10.1016/j.neucom.2015.05.080
0925-2312/

coupled neurons. To describe the complicated interaction between billions of neurons in a large neural network, the neurons are generally lumped into highly connected sub-networks [32]. These sub-network interactions (pathological synchronization) may bring about serious problems such as Parkinsons disease, essential tremor, and epilepsy [33,34].

Inspired by the above works, we consider in this paper a two-level hierarchical system, which consists of two coupled modules of interacting nonlinear neuron oscillators with time delays and $\mathbb{Z}_n-$symmetry. This network is depicted schematically in Fig. 1, and given by the following system of delay differential equations (DDEs):

$$\begin{cases} \dot{x}_{0j}(t) = -x_{0j}(t) + f(x_{0j}(t-\tau)) + g(x_{0,j+1}(t-\tau)) + h(x_{1j}(t-\tau)), \\ \dot{x}_{1j}(t) = -x_{1j}(t) + f(x_{1j}(t-\tau)) + g(x_{1,j+1}(t-\tau)) + h(x_{0j}(t-\tau)), \end{cases} \tag{1.2}$$

where $j = 0, 1, \ldots, n-1 (\mathrm{mod}\ n)$, the activation functions $f, g, h \in C^1(\mathbb{R}; \mathbb{R})$ satisfying $f(0) = g(0) = h(0) = 0$. In this model, the individual elements are represented by a scalar equation, composed of a linear decay term and a nonlinear, time delayed self-feedback.

In model (1.2), each sub-network can be considered as an identical ring module in which n elements are coupled in such a way that the invariance under cyclic permutations is attained. Noticing that the decay rate μ in system (1.1) can be normalized through the time transformation, system (1.2) is a natural extension of system (1.1). System (1.2) is also a particularly simple example of a symmetric system exhibiting a hierarchical structure with two levels: a "macro" level concerning the interactions between the groups and a "micro" level concerning the interactions within the groups. The overall symmetry of system (1.2) can then be represented as a product of permutation groups, $\mathbb{Z}_2 \times \mathbb{Z}_n$, which allows us to study the dynamics analytically. Furthermore, the symmetry implies generally a certain spatial invariant of the dynamical systems. For example, the spatio-temporal patterns of bifurcating periodic solutions can be characterized precisely according to symmetric Hopf bifurcation theory, which is due to the pioneering work of Wu [35] (based on the topological methods and theorem by Golubitsky [36]).

The main difference from the models considered in [30,31] is that model (1.2) is a large-scale network, each sub-network of which is composed of arbitrary n neurons with an unidirectional ring structure. This adds some complications to the analysis and computation but, as we shall show, also allows for more interesting dynamics of the system. In this paper, we are interested in studying how the time delay can affect the stability of two-level hierarchical system (1.2). We are concerned about the occurrence of bifurcating periodic solutions when the delay τ passes through a critical value, and spatio-temporal patterns of the bifurcating periodic oscillations depending on the $\mathbb{Z}_2 \times \mathbb{Z}_n-$symmetry. In addition, the stability of these periodic solutions is clearly important in applications, but also poses significant computational challenges. We also manage to obtain some formula about the direction of the Hopf bifurcation and the stability of the bifurcated periodic solution by using normal form method and center manifold introduced by Faria and Magalháes [37,38].

The outline of this paper is as follows. In Section 2, we discuss the linear stability of the equilibrium by analyzing the distribution of roots of the associated characteristic equation. The local Hopf bifurcation and spatio-temporal patterns of it are addressed in Section 3. Section 4 is devoted to the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions. An example and numerical simulations are presented to illustrate the results in Section 5. Finally, a brief discussion is drawn in Section 6.

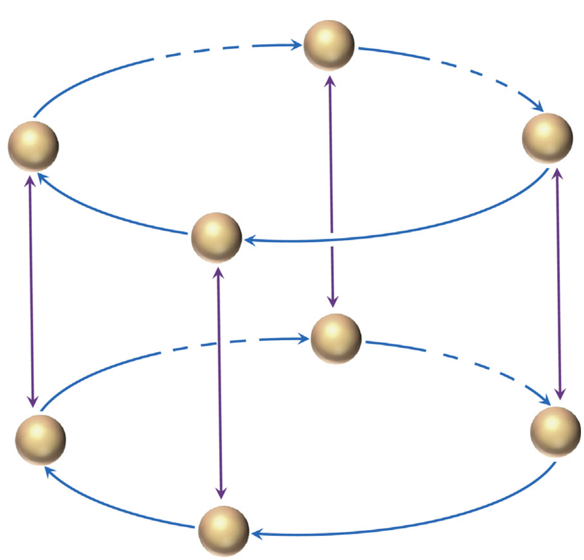

Fig. 1. Architecture of model (1.2).

2. Linear stability analysis

It is obvious that system (1.2) admits the trivial solution $\hat{x} = 0$. The linearization of (1.2) at the origin is given by

$$\begin{cases} \dot{x}_{0j}(t) = -x_{0j}(t) + a x_{0j}(t-\tau) + b x_{0,j+1}(t-\tau) + c x_{1j}(t-\tau), \\ \dot{x}_{1j}(t) = -x_{1j}(t) + a x_{1j}(t-\tau) + b x_{1,j+1}(t-\tau) + c x_{0j}(t-\tau), \end{cases} \tag{2.1}$$

where j mod n, and $a = f'(0)$, $b = g'(0)$, $c = h'(0)$.

Letting $x(t) = (x_{00}(t), x_{01}(t), \ldots, x_{0,n-1}(t), x_{10}(t), x_{11}(t), \ldots, x_{1,n-1}(t))^{\mathrm{T}} \in \mathbb{R}^{2n}$ and $x_t(\theta) = x(t+\theta) \in C([-\tau, 0], \mathbb{R}^{2n})$, system (2.1) can be rewritten as

$$\dot{x}(t) = L_\tau x_t, \tag{2.2}$$

where the linear operator $L_\tau : C([-\tau, 0], \mathbb{R}^{2n}) \to \mathbb{R}^{2n}$ is given by

$$L_\tau \varphi = -\varphi(0) + M\varphi(-\tau), \tag{2.3}$$

$M = \mathrm{circ}(M_1, M_2)$ is a circle block matrix, $M_1 = \mathrm{circ}(a, b, 0, \ldots, 0)$ is a circulant matrix of order n, and $M_2 = c\,\mathrm{Id}_n$, Id_n denotes the identity matrix of order n. It is well-known that for each fixed delay τ, the linear system (2.1) generates a strongly continuous semigroup of linear operators with the infinitesimal generator $A(\tau)$ given by

$$A(\tau)\varphi = \dot{\varphi}, \quad \varphi \in \mathrm{Dom}(A(\tau)),$$
$$\mathrm{Dom}(A(\tau)) = \{\varphi \in C([-\tau, 0], \mathbb{R}^{2n});\ \dot{\varphi} \in C([-\tau, 0], \mathbb{R}^{2n}),\ \dot{\varphi}(0) = L_\tau \varphi\}. \tag{2.4}$$

We recall that $A(\tau)$ has only the point spectrum, and the spectrum $\sigma(A(\tau))$ consists of eigenvalues which are solutions of the characteristic equation

$$\det \Delta(\tau, \lambda) = 0, \tag{2.5}$$

where the characteristic matrix is given by

$$\Delta(\tau, \lambda) = \lambda\, \mathrm{Id}_{2n} - L_\tau(\mathrm{e}^{\lambda \cdot} \mathrm{Id}_{2n})$$
$$= (\lambda + 1)\mathrm{Id}_{2n} - M\mathrm{e}^{-\lambda\tau}, \quad \lambda \in \mathbb{C}.$$

Let $\chi = \mathrm{e}^{\mathrm{i}2\pi/n}$, $v_q = (1, \chi^q, \ldots, \chi^{(n-1)q})^{\mathrm{T}}$, and $v_{pq} = (v_q, (-1)^p v_q)^{\mathrm{T}}$. Noting that $Mv_{pq} = [a + (-1)^p c + b\chi^q] v_{pq}$, we have

$$\Delta(\tau, \lambda) v_{pq} = \left\{\lambda + 1 - [a + (-1)^p c + b\chi^q]\mathrm{e}^{-\lambda\tau}\right\} v_{pq}. \tag{2.6}$$

Hence, we obtain

$$\det\Delta(\tau, \lambda) = \prod_{q=0}^{n-1} \prod_{p=0}^{1} \Delta_{pq}, \tag{2.7}$$