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Comparison principles and stability of nonlinear fractional-order cellular neural networks with multiple time delays

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ABSTRACT

Some comparison principles for fractional-order linear systems with multiple time delays are established, after Mittag–Leffler functions are showed to be positive. Then by the stability theory of fractional linear delayed systems, the comparison system with multiple time delays is showed to be asymptotically stable under some conditions. Based on the comparison results, the asymptotical stability of the original systems follows from that of the comparison system. Then the obtained results are applied to investigate the asymptotical stability of nonlinear fractional-order cellular neural networks with multiple time delays. In terms of the inequality satisfied by the fractional derivative of Lyapunov function, some criteria ensuring asymptotical stability of fractional neural models are derived. Numerical simulations are presented to demonstrate the validity and feasibility of the proposed stability criteria.

1. Introduction

As we know, neural networks have been one of the most extensively investigated topics in many research and application areas in solving various classes of engineering problems, such as image and signal processing, associative memory, pattern recognition, parallel computation, control and optimization and so on [1–7]. Equilibrium and stability properties of neural networks are of great importance in the design of dynamical neural networks. Frequently, time delays will be encountered in biological and artificial neural networks [8–12], and their existence will often cause complex behaviors, such as oscillation and instability [13,14]. So delayed models are more general in practical applications of neural networks. Now fractional calculus has already been introduced into neural models by many researchers.

Fractional calculus is believed to have stemmed from a question raised in the year 1695. It is the generalization of integer-order calculus to arbitrary order one. It is commonly called fractional-order calculus, including fractional-order derivatives and fractional-order integrals. Compared with the classical integer-order models, fractional-order models provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. It would be far better if many practical problems are described by fractional-order dynamical systems rather than integer-order ones. So many systems could be more elegantly described by fractional calculus, such as viscoelastic systems, dielectric polarization, electromagnetic waves, heat conduction, robotics, biological systems, finance and so on, see [15–21].

Nowadays, some stability results and chaos about fractional neural models have been derived. For instance, in [22], stability, multi-stability bifurcations and chaos of fractional-order neural networks of Hopfield type were investigated. In [23], the dynamics of noninteger order cellular neural networks were introduced. In [24], stability of fractional-order autonomous neural networks was described by handling a new fractional-order differential inequality. In [25] Huang et al. channeled their energy into chaos and hyperchaos in fractional-order cellular neural networks, and so on.

And there are also many researchers paying their attentions to fractional-order neural networks with delay. Such as in [26], a sufficient condition was established for the uniform stability of fractional-order neural networks with delay. In [27], the robust exponential synchronization problem of a class of chaotic delayed neural networks with different parametric uncertainties was studied. In [28], the global stability analysis of fractional-order Hopfield neural networks with time delay was investigated. Based on the stability theory of linear fractional systems with time delays, Ref. [29] investigated the synchronization between fractional chaotic systems with delay. Note that the models in [28,29] admit the single time delay. However, due to the difficulty in dealing with stability of nonlinear fractional systems with delays, there are few works on the stability of nonlinear fractional neural networks with multiple time delays.

In this paper, Mittag–Leffler functions are first showed to be positive. Based on that, some comparison principles for linear

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fractional systems with multiple time delays are presented. Note from the comparison principle that the stability of the original systems will follow from that of the comparison system. Using the obtained results, stability of nonlinear fractional-order cellular neural networks with multiple time delays is investigated. By constructing Lyapunov functions, some conditions ensuring stability are obtained in terms of the inequality satisfied by the fractional derivative of Lyapunov functions.

The paper is organized as follows. In Section 2, some preliminaries about fractional calculus and some lemmas are presented. In Section 3, comparison principles for fractional-order systems with multiple time delays are derived. In Section 4, stability criteria of fractional-order cellular neural networks with multiple time delays are given. In Section 5, numerical simulations are performed to show the effectiveness of theoretical analysis. Finally, some conclusions are drawn in Section 6.

2. Preliminaries

In fractional calculus, the traditional definitions of the integral and derivative of a function are generalized from integer orders to arbitrary ones. In the time domain, the fractional-order derivative and integral operators are defined by a Laplace convolution operation as follows.

Definition 1 (*Kilbas et al.* [20, p. 92]). Caputo fractional derivative with order α for function $x(t)$ is defined as

$$^{C}D_{t_0}^{\alpha}x(t)=\frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}x^{(m)}(\tau)\,d\tau,$$

where $0\le m-1\le\alpha<m$, $m\in Z_+$, and $t=t_0$ is the initial time, $\Gamma(\cdot)$ is the Gamma function.

Definition 2 (*Kilbas et al.* [20, p. 69]). Riemann–Liouville fractional integral of order $\alpha>0$ for a function f: $R\to R$ is defined as

$$I_{t_0}^{\alpha}f(t)=\frac{1}{F(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau)\,d\tau,$$

where $t=t_0$ is the initial time, $\Gamma(\cdot)$ is the Gamma function.

Definition 3 (*Kilbas et al.* [20, p. 70]). Riemann–Liouville fractional derivative with order α for function x: $R\to R$ is defined as

$$^{RL}D_{t_0}^{\alpha}x(t)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}x(\tau)\,d\tau,$$

where $0\le m-1\le\alpha<m$, $m\in Z_+$, and $t=t_0$ is the initial time, $\Gamma(\cdot)$ is the Gamma function.

The Laplace transform of the Caputo fractional-order derivative is

$$\mathcal{L}(^{C}D_{0}^{\alpha}x(t))=s^{\alpha}\mathcal{L}(x(t))-\sum_{k=0}^{n-1}s^{\alpha-k-1}x^{(k)}(0),$$

where $\alpha>0$, $n=[a]+1$. Since the initial conditions of Laplace transform of Caputo derivatives take the same forms as those in classic integer-order cases, the Caputo derivative is employed in this paper.

Lemma 1 (*Aguila-Camacho et al.* [30]). *Let $x(t)\in R^n$ be a continuous and derivable function. Then, for any time instant $t\ge t_0$*

$$\tfrac{1}{2}\,^{C}D_{t_0}^{\alpha}x^2(t)\le x(t)\,^{C}D_{t_0}^{\alpha}x(t),\quad\forall\alpha\in(0,1).$$

Lemma 2. *Consider the following fractional-order differential equation:*

$$\left\{\,^{C}D_{0,t}^{\alpha}x(t)=-\beta x(t)x_0=x(0)=1,\right. .\tag{1}$$

If $\beta>0$, then $x(t)>0$ for all $t\ge0$, that is $E_\alpha(-\beta t^\alpha)>0$ for all $t\ge0$.

Proof. From Theorem 3.25 in [20, p. 202], we have $x(t)$ is continuous. Using Reductio ad absurdum, assume that t_1 is the first point such that $x(t)=0$, it means that when $t<t_1$, $x(t)>0$; when $t=t_1$, $x(t)=0$. Then we have $[x(t)]'_{t=t_1^-}<0$. One obtains

$$\begin{aligned}^{C}D_{0,t_1}^{\alpha}x(t)&=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-\tau)^{-\alpha}[x(\tau)]'\,d\tau\\&=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-\tau)^{-\alpha}\,dx(\tau)\\&=\frac{x(\tau)(t_1-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\bigg|_{\tau=0}^{\tau=t_1}-\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-\tau)^{-\alpha-1}x(\tau)\,d\tau,\end{aligned}\tag{2}$$

where

$$\begin{aligned}\frac{x(\tau)(t_1-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\bigg|_{\tau=0}^{\tau=t_1}&=\lim_{\tau\to t_1}\frac{x(\tau)}{(t_1-\tau)^{\alpha}\Gamma(1-\alpha)}-\frac{1}{t_1^{\alpha}\Gamma(1-\alpha)}\\&=\lim_{\tau\to t_1}\frac{x'(\tau)(t_1-\tau)^{1-\alpha}}{\alpha\Gamma(1-\alpha)}-\frac{1}{t_1^{\alpha}\Gamma(1-\alpha)}\\&=-\frac{1}{t_1^{\alpha}\Gamma(1-\alpha)}.\end{aligned}\tag{3}$$

Submitting (3) into (2), one obtains

$$\begin{aligned}^{C}D_{0,t_1}^{\alpha}x(t)&=-\frac{1}{t_1^{\alpha}\Gamma(1-\alpha)}-\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-\tau)^{-\alpha-1}x(\tau)\,d\tau\\&=-\frac{1}{t_1^{\alpha}\Gamma(1-\alpha)}-\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{t_1^-}(t_1-\tau)^{-\alpha-1}x(\tau)\,d\tau<0.\end{aligned}\tag{4}$$

On the other hand, $^{C}D_{0,t_1}^{\alpha}x(t)=-\beta x(t_1)=0$. That is a contradiction, so $x(t)>0$, namely, $E_\alpha(-\beta t^\alpha)>0$, since $E_\alpha(-\beta t^\alpha)$ is the unique solution of Cauchy problem (1) (see, for example, Ref. [20]). This completes the proof.□

Then we consider the following Riemann–Liouville fractional-order differential equation:

$$\left\{\,^{RL}D_{0,t}^{\alpha}x(t)=-\lambda x(t),I_0^{1-\alpha}x(0+)=b>0,\right. .\tag{5}$$

where $0<\alpha<1$, $\lambda>0$, $x(t)\in R^1$. From Theorem 4.1 in [20], Eq. (5) has a unique solution $x(t)=bt^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$.

Lemma 3. *Supposing that $x(t)$ is the solution of Eq. (5), then $x(t)>0$, that is, $E_{\alpha,\alpha}(-\lambda t^\alpha)>0$.*

Proof. Note that the solution $x(t)$ is continuous and differentiable in $(0,+\infty)$. Using Reductio ad absurdum, assume that t_1 is the first time such that $x(t)=0$. It means that when $t<t_1$, $x(t)>0$; when $t=t_1$, $x(t)=0$. Since $^{RL}D_{0,t}^{\alpha}x(t)=(d/dt)I_0^{1-\alpha}x(t)$, let $f(t)=I_0^{1-\alpha}x(t)$, then (5) can be written into

$$f'(t)=-\lambda x(t),\quad f'(t_1^-)=0.$$

Let $\delta>0$ be small enough, we have

$$\begin{aligned}\Gamma(1-\alpha)(f(t_1-\delta)-f(t_1))&=\int_0^{t_1-\delta}(t_1-\delta-\tau)^{-\alpha}x(\tau)\,d\tau\\&-\int_0^{t_1}(t_1-\tau)^{-\alpha}x(\tau)\,d\tau.\end{aligned}\tag{6}$$

Let $\tau'=\delta+\tau$, then one has

$$\int_0^{t_1-\delta}(t_1-\delta-\tau)^{-\alpha}x(\tau)\,d\tau=\int_\delta^{t_1}(t_1-\tau')^{-\alpha}x(\tau'-\delta)\,d\tau'.\tag{7}$$