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Letters

Dynamics in fractional-order neural networks

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ABSTRACT

This paper investigates a general class of neural networks with a fractional-order derivative. By using the contraction mapping principle, Krasnoselskii fixed point theorem and the inequality technique, some new sufficient conditions are established to ensure the existence and uniqueness of the nontrivial solution. Moreover, uniform stability of the fractional-order neural networks is proposed in fixed time-intervals. Finally, some examples are given to illustrate the effectiveness of theoretical results.

1. Introduction

Neural networks have received much attention in recent years [1–14]. Ranging from image processing, combinatorial optimization, associative memories, pattern recognition and other areas, neural networks have witnessed a large amount of successful applications in many fields. It is well known that the unique globally stable equilibrium is crucial to solve some optimization problems. Therefore, the existence, uniqueness and stability of the equilibrium for neural networks have received considerable interests and various kinds of conditions have been derived [8–10].

As an old topic, the study of fractional-order calculus has re-attracted tremendous attention of many researchers [15,16] because it has proved to be valuable tools in the modeling of many physics and engineering phenomena. Fractional-order calculus provides a powerful tool for describing memory and hereditary properties of the systems where such effects are neglected or difficult to describe to the integer order models. The main advantage of fractional-order systems is that the fractional-order differential operator is nonlocal in the sense that it takes into account the fact that the future state not only depends upon the present state but also upon all the history of its previous states, while an integer-order differential operator is a local operator.

In [17], Lundstrom et al. pointed out that fractional-order recurrent neural networks are expected to be very effective in many applications such as parameter estimation due to its memory and hereditary properties. Therefore, it is interesting and challenging to investigate the dynamical behavior of fractional-order neural networks.

Recently, some important and interesting results on fractional-order neural networks have been obtained and various issues have been investigated [18–26] by many authors. In [21], Huang et al. proposed a fractional-order cellular neural networks and discussed the complex dynamics of such a system by numerical simulations. In [22], Kaslik and Sivasundaram showed that fractional-order neural networks can exhibit rich dynamical behavior such as multi-stability, bifurcations and chaos by applying the linear stability theory of fractional-order system. Yu et al. investigated α-stability and α-synchronization for fractional-order neural networks by introducing a new fractional-order differential inequality. In addition, the existence and α-stability of equilibrium point were considered in [23]. In [24], Chen et al. obtained a sufficient condition for uniform stability of a class of fractional-order delayed neural networks. Finite-time stability of a class of fractional-order neural networks is studied in [25] by using Laplace transform, the generalized Gronwall inequality and Mittag-Leffler functions. In [26], the authors have investigated the finite-time stability of Caputo fractional distributed delayed neural networks by applying the fractional calculus theory and a generalized Gronwall–Bellman inequality technique.

However, to the best of our knowledge, the existence of nontrivial solution has not been studied in the above literature. In fact, nontrivial solutions' existence should not be ignored when

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discussing stability of the equilibrium point or synchronization behavior of fractional-order neural networks are discussed. Motivated by the above discussions, this paper considers a general class of fractional-order neural networks described by

$$\begin{cases} {}^{c}D^{\alpha}x_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + I_i, \quad t \geq 0, \\ x_i(0) = x_{i0}, \end{cases} \tag{1}$$

where $0<\alpha<1$, ${}^{c}D^{\alpha}$ is the Caputo fractional derivative, n corresponds to the number of the units in a neural network, $x_i(t)$ stands for the state variable of the ith neuron at the time t, $f_j(x_j(t))$ denotes the measure of activation to its incoming potentials of the unit jth at time t, $c_i>0$ $(i=1,\dots,n)$, I_i is the constant control input vector and a_{ij} represents the strength of the neuron interconnection within the network.

The paper is organized as follows. In Section 2, some preliminaries on the fractional-order calculus are reminded. In Section 3, we present some new sufficient conditions to ensure the existence, uniqueness of the nontrivial solution and also uniform stability of the fractional-order neural networks is obtained in a finite time interval. In Section 4, the efficiency of the proposed theorems is illustrated by two examples. Some conclusions are presented in Section 5.

2. Preliminaries

Some preliminaries regarding fractional calculus are presented in this section. For more details, refer to [27].

There are several definitions for fractional derivatives such as Grünwald–Letnikov, Riemann–Liouville and Caputo. Throughout the paper, the Caputo definition is used since its initial conditions take the same form as the integer order differential equation.

Definition 1 (*See Podlubny [27]*). The Riemann–Liouville fractional integral operator of order $\alpha>0$ of a function h is defined by

$$I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds,$$

where Γ is the gamma function.

Definition 2 (*See Podlubny [27]*). The Caputo fractional-order derivative of order $\alpha>0$ of a function $h(t)$ is defined by

$${}^{c}D^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)\, ds, \quad n-1<\alpha<n, \ n\in\mathbb{N}^+.$$

In the sequel, the following lemma is needed.

Lemma 1 (*See Zhang [28]*). *Let $\alpha>0$, then the differential equation*

${}^{c}D^{\alpha}y(t)=h(t)$,

has solutions $y(t)=I^{\alpha}h(t)+c_0+c_1 t+c_2 t^2+\cdots+c_{n-1}t^{n-1}$, where $c_i\in R$, $n=[\alpha]+1$.

Lemma 2 (*Krasnoselskii lemma (see [29])*). *Let D be a closed convex and nonempty subset of a Banach space X. Let ϕ_1,ϕ_2 be the operators such that*

(i) *$\phi_1 x+\phi_2 y\in D$ whenever $x,y\in D$;*
(ii) *ϕ_1 is compact and continuous;*
(iii) *ϕ_2 is a contraction mapping.*

Then, there exists $x\in D$ such that $\phi_1 x+\phi_2 x=x$.

As a consequence of Lemma 1, we define the solution of system (1).

Definition 3. The continuous function $x_i(t)$ is said to be a solution of the system (1) if the following condition

$$x_i(t) = x_{i0} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[-c_i x_i(s) + \sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + I_i \right] ds$$

is satisfied.

3. Existence, uniqueness and uniform stability of solution

In this section, we will consider the existence and uniqueness of solution to system (1) by using the fixed point theorem. To establish our main results, it is necessary to make the following assumption.

Assumption 1. The neuron activation functions f_i $(i=1,\dots,n)$ satisfy the Lipschitz condition, i.e. there exist nonnegative constants l_i such that for any $u,v\in R$

$$|f_i(u)-f_i(v)| \leq l_i|u-v| \leq \max_{1\leq i\leq n} l_i|u-v| = l_0|u-v|.$$

Let $X=\{x \mid x=(x_1,x_2,\dots,x_n)^T,\ x_i\in C[0,T]\}$, it is easy to see that X is a Banach space with the norm $\|x\| = \sup_{0\leq t\leq T}(\sum_{j=1}^{n}|x_i(t)|^p)^{1/p}$.

Now, we are ready to present the first result.

Theorem 1. *Under Assumption 1, system (1) has a unique solution on $[0,T]$, if there exists a real number $p>1$ such that*

$$c_0 T^{\alpha} n^{1/p} + l_0 T^{\alpha} \left(\sum_{i=1}^{n} \xi_i^p \right)^{1/p} < \Gamma(\alpha+1), \tag{2}$$

where $\xi_i = [\sum_{j=1}^{n}|a_{ij}|^{p/(p-1)}]^{(p-1)/p}$, $c_0=\max_{1\leq i\leq n} c_i$.

Proof. Define $F: X\to X$ as

$(Fx)(t) = ((Fx_1)(t),(Fx_2)(t),\dots,(Fx_n)(t))^T$,

where

$$(Fx_i)(t) = x_{i0} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[-c_i x_i(s) + \sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + I_i \right] ds.$$

To make the readers easy to follow, the proof is divided into two steps as follows. □

Step 1: Firstly, we prove $FB_\delta\subset B_\delta$, where $B_\delta=\{x\in X: \|x\|\leq\delta\}$. Let

$$\delta \geq \frac{\Gamma(\alpha+1)[\sum_{i=1}^{n}|x_{i0}|^p]^{1/p} + I_0 T^{\alpha} n^{1/p} + f_0 T^{\alpha}[\sum_{i=1}^{n} a_{i0}^p]^{1/p}}{\Gamma(\alpha+1) - c_0 T^{\alpha} n^{1/p} - l_0 T^{\alpha}(\sum_{i=1}^{n}\xi_i^p)^{1/p}}.$$

where $I_0=\max_{1\leq i\leq n}|I_i|$, $f_0=\max_{1\leq i\leq n}|f_j(0)|$, $a_{i0}=\sum_{j=1}^{n}|a_{ij}|$.

Minkowski inequality gives that

$$\left[\sum_{i=1}^{n} (a_i+b_i+\cdots+l_i)^p \right]^{1/p} \leq \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} + \cdots + \left(\sum_{i=1}^{n} l_i^p \right)^{1/p},$$

$a_i,b_i,\dots,l_i\geq 0$, $p>1$, $i=1,2,\dots,n$.

Moreover

$$\begin{aligned} \|Fx\| &= \sup_{0\leq t\leq T} \left\{ \sum_{i=1}^{n} \left| x_{i0} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[-c_i x_i(s) + \sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + I_i \right] ds \right|^p \right\}^{1/p} \\ &\leq \left[\sum_{i=1}^{n} |x_{i0}|^p \right]^{1/p} + \sup_{0\leq t\leq T} \left[\sum_{i=1}^{n} \left(\int_0^t \frac{(t-s)^{\alpha-1} c_i |x_i(s)|}{\Gamma(\alpha)} ds \right)^p \right]^{1/p} \\ &\quad + \sup_{0\leq t\leq T} \left[\sum_{i=1}^{n} \left(\int_0^t \frac{(t-s)^{\alpha-1}|I_i|}{\Gamma(\alpha)} ds \right)^p \right]^{1/p} \end{aligned}$$