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Optimal time-convex hull for a straight-line highway in L_p-metrics ☆

Bang-Sin Dai [b], Mong-Jen Kao [a,*], D.T. Lee [a]

[a] *Institute of Information Science, Academia Sinica, Taiwan*
[b] *Department of Computer Science and Information Engineering, National Taiwan University, Taiwan*

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ABSTRACT

We consider the problem of computing the time-convex hull of a point set under the general L_p metric in the presence of a straight-line highway in the plane. The traveling speed along the highway is assumed to be faster than that off the highway, and the shortest time-path between a distant pair may involve traveling along the highway. The time-convex hull TCH(P) of a point set P is the smallest set containing both P and *all* shortest time-paths between any two points in TCH(P). In this paper we give an algorithm that computes the time-convex hull under the L_p metric in optimal $\mathcal{O}(n \log n)$ time for a given set of n points and a real number p with $1 \le p \le \infty$.

1. Introduction

Path planning, in particular, shortest time-path planning, in complex transportation networks has become an important yet challenging issue in recent years. With the usage of heterogeneous moving speeds provided by different means of transportation, the *time-distance* between two points, i.e., the amount of time it takes to go from one point to the other, is often more important than their straight-line distance. With the reinterpretation of distances by the time-based concept, fundamental geometric problems such as convex hull, Voronoi diagrams, facility location, etc., have been reconsidered recently in depth and with insights [8,5,1].

From the theoretical point of view, straight-line highways which provide faster moving speed and which we can enter and exit at any point is one of the simplest transportation models to explore. The speed at which one can move along the highway is assumed to be $v > 1$, while the speed off the highway is 1. Generalization of convex hulls in the presence of highways was introduced by Hurtado et al. [9], who suggested that the notion of convexity be defined by the inclusion of shortest time paths, instead of straight-line segments, i.e., a set S is said to be **convex** if it contains the shortest time-path between any two points of S. Using this new definition, the time-convex hull TCH(P) for a set P is the closure of P with respect to the inclusion of shortest time-paths.

In following work, Palop [13] studied the structure of TCH(P) in the presence of a highway and showed that it is composed of convex polygons (clusters) possibly with a segment of the highway connecting all the components. A particularly interesting fact implied by the hull structure is that, the shortest time-path between each pair of inter-cluster points must contain a piece of traversal along the highway, while similar assertions do not hold for intra-cluster pairs of points: A distant

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* Corresponding author.
E-mail addresses: f94922074@ntu.edu.tw (B.-S. Dai), mong@iis.sinica.edu.tw (M.-J. Kao), dtlee@ieee.org (D.T. Lee).

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pair of points (p, q) whose shortest time-path contains a segment of the highway could still belong to the same cluster, for there may exist other points from the same cluster whose shortest time-path to either p or q does not use the highway at all. This suggests that, the structure of TCH(P) in some sense indicates the degree of convenience provided by the underlying transportation network. We prefer clusters of higher densities, i.e., any cluster with a large ratio between the number of points of P it contains and the area of that cluster. For sparse clusters, we may want to break them and benefit distant pairs they contain by enhancing the transportation infrastructure.

The approach suggested by Palop [13] for the presence of a highway involves enumeration of shortest time-paths between all pairs of points and hence requires $\Theta\left(n^2\right)$ time, where n is the number of points. This problem was later studied by Yu and Lee [14], who proposed an approach based on incremental point insertions in a highway-parallel monotonic order. However, the proposed algorithm does not return the correct hull in all circumstances as particular cases were overlooked. The first sub-quadratic algorithm was given by Aloupis et al. [3], who proposed an $\mathcal{O}\left(n\log^2 n\right)$-time algorithm for the L_2 metric and an $\mathcal{O}(n\log n)$-time algorithm for the L_1 metric, following the incremental approach suggested by [14] with careful case analysis. To the best of our knowledge, no previous results regarding metrics other than L_1 and L_2 were presented.

Our focus and contribution In this paper we address the problem of computing the time-convex hull of a point set in the presence of a straight-line highway under the L_p metric for a given real number p with $1 \le p \le \infty$. The contribution of this paper consists of the following two parts.

First, we adopt the concept of *wavefront propagation*, a notion commonly used for path planning [8,2], and derive basic properties required for depicting the hull structure under the general L_p metric. When the shortest path between two points is not uniquely defined, e.g., in L_1 and L_∞ metrics, we propose a re-evaluation of the existing definition of convexity. Previous works concerning convex hulls under metrics other than L_2, e.g., Ottmann et al. [12] and Aloupis et al. [3], assume a particular path to be taken when multiple choices are available. However, this assumption allows the boundary of a convex set to contain reflex angles, which in some sense deviates from the intuition of a set being convex.

In this work we adopt the definition that requires a convex set to include *every shortest path* between any two points it contains. This ensures that no reflex angles are formed on the boundary of any set being convex. Although this definition fundamentally simplifies the shapes of convex sets for L_1 and L_∞ metrics, we show that the nature of the problem is not altered when time-based concepts are considered. In particular, the problem of deciding whether any pair of the given points belong to the same cluster under the L_p metric requires $\Omega(n\log n)$ time under the algebraic computation model [6], for all $1 \le p \le \infty$.

Second, we provide an optimal $\mathcal{O}(n\log n)$-time algorithm for computing the time-convex hull for a given set of points. The known algorithm due to Aloupis et al. [3] stems from a scenario in the cluster-merging step where we have to check for the existence of intersections between a line segment and a set of convex curves composed of parabolae and line segments, which leads to their $\mathcal{O}\left(n\log^2 n\right)$-time algorithm. In our paper, we tackle this situation by making an observation on the duality of cluster-merging conditions and reduce the problem to the geometric query of deciding if any of the given points lies above a line segment of an arbitrary slope. This approach greatly simplifies the algorithmic structure and can be easily generalized to other L_p-metrics for $1 \le p \le \infty$. For this particular geometric problem, we use a data structure due to Mitchell [11] to answer this query in logarithmic time. All together this yields our $\mathcal{O}(n\log n)$-time algorithm. We remark that, although our adopted definition of convexity simplifies the shape of convex sets under the L_1 and the L_∞ metrics, the algorithm we propose does not take advantage of this specific property and also works for the original notion for which only a particular path is to be included.

Organization of this paper The rest of this paper is organized as follows. In Section 2 we formally introduce our problem and define related concepts. In Section 3, we present our optimal $\mathcal{O}(n\log n)$-time algorithm for the L_2-metric. In Section 4 we discuss the problem of path-planning under general L_p-metrics and give a characterization of shortest time-paths in the presence of straight-line highways. In Section 5 we derive properties related to the walking regions and the time-convex hulls that are necessary for our optimal algorithm. In Section 6 we prove an $\Omega(n\log n)$ lower bound on the time complexity of this problem, followed by extending our optimal algorithm for the L_2-metric to general L_p-metrics. We conclude in Section 7 with some remarks and overview on future research directions.

2. Preliminaries

In this section, we give precise definitions of the notions as well as sketches of previously known properties that are essential to our work. We begin with the general L_p distance metric and basic time-based concepts.

Definition 1 *(Distance in the L_p-metrics).* For any real number $p \ge 1$ and any two points $q_i, q_j \in \mathbb{R}^n$ with coordinates $(i_1, i_2, \ldots, i_n)$ and $(j_1, j_2, \ldots, j_n)$, the distance between q_i and q_j under the L_p-metric is defined to be $d_p\left(q_i, q_j\right) = \left(\sum_{k=1}^n |i_k - j_k|^p\right)^{\frac{1}{p}}$.