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Algorithms and bounds for drawing non-planar graphs with crossing-free subgraphs ☆

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A B S T R A C T

We initiate the study of the following problem: *Given a non-planar graph G and a planar subgraph S of G, does there exist a straight-line drawing Γ of G in the plane such that the edges of S are not crossed in Γ by any edge of G?* We give positive and negative results for different kinds of connected spanning subgraphs S of G. Moreover, in order to enlarge the subset of instances that admit a solution, we consider the possibility of bending the edges of G not in S; in this setting we discuss different trade-offs between the number of bends and the required drawing area.

1. Introduction

Many papers in graph drawing address the problem of computing drawings of non-planar graphs with the goal of mitigating the negative effect that edge crossings have on the readability of the drawing. Several of these papers describe crossing minimization methods, which are effective and computationally feasible for relatively small and sparse graphs (see [9] for a survey). Other papers study how non-planar graphs can be drawn such that the "crossing complexity" of the drawing is somewhat controlled, either in the number or in the type of crossings. They include the study of *k-planar drawings*, in which each edge is crossed at most k times (see, e.g., [8,14,15,18,21,26,30]), of *k-quasi planar drawings*, in which no k pairwise crossing edges exist (see, e.g., [1,2,13,19,29,33]), and of *large angle crossing drawings*, in which any two crossing edges form a sufficiently large angle (see [17] for a survey). Most of these drawings exist only for sparse graphs.

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In this paper we introduce a new graph drawing problem concerned with the drawing of non-planar graphs. Namely: *Given a non-planar graph G and a planar subgraph S of G, decide whether G admits a drawing Γ such that (in Γ) the edges of S are not crossed by any edge of G. Compute Γ if it exists.*

Besides its intrinsic theoretical interest, this problem is also of practical relevance in many application domains. Indeed, distinct groups of edges in a graph may have different semantics, and a group can be more important than another for some applications; in this case a visual interface might attempt to display more important edges without intersections. Furthermore, the user could benefit from a layout in which a connected spanning subgraph is drawn crossing free, since it would support the user to quickly recognize paths between any two vertices, while keeping the other edges of the graph visible.

Please note that the problem of recognizing specific types of subgraphs that are not self-crossing (or that have few crossings) in a given drawing Γ, has been previously studied (see, e.g., [23,25,28,31]). This problem, which turns out to be NP-hard for most different kinds of instances, is also very different from our problem. Indeed, in our setting the drawing is not the input, but the output of the problem. Also, we require that the given subgraph S is not crossed by any edge of the graph, not only by its own edges.

In this paper we concentrate on the case in which S is a connected spanning subgraph of G and consider both straight-line and polyline drawings of G. Namely:

(i) In the straight-line drawing setting we prove that if S is any given spanning spider or caterpillar, then a drawing of G where S is crossing free always exists; such a drawing can be computed in linear time and requires polynomial area (Section 3.1), although our construction for caterpillars does not compute integer coordinates. We also show that this positive result cannot be extended to any spanning tree, but we describe a large family of spanning trees that always admit a solution, and we observe that any graph G contains such a spanning tree; unfortunately, our drawing technique for this family of trees may require exponential area. Finally, we characterize the instances $\langle G, S\rangle$ that admit a solution when S is a triconnected spanning subgraph, and we provide a polynomial-time testing and drawing algorithm, whose layouts have polynomial area (Section 3.2).

(ii) We investigate polyline drawings where only the edges of G not in S are allowed to bend. In this setting, we show that all spanning trees can be realized without crossings in a drawing of G of polynomial area, and we describe efficient algorithms that provide different trade-offs between the number of bends per edge and the required drawing area (Section 4). Also, we consider the case in which S is any given biconnected spanning subgraph. In this case, we provide a characterization of the positive instances, which yields drawings with polynomial area, if only one bend per edge is allowed.

We finally remark that the study of our problem has been receiving some interest in the graph drawing community. In particular, Schaefer proved that given a graph G and a planar subgraph S of G, testing whether there exists a polyline drawing of G where the edges of S are never crossed can be done in polynomial time [32]. This result was shortly afterwards improved to linear time in a work by Da Lozzo and Rutter [10], who studied this problem in the framework of streamed graph drawing. In these two works, differently from ours, there is no restriction on the number of bends per edge and the edges of S are not required to be drawn as straight-line segments.

In Section 2 we give some preliminary definitions that will be used in the rest of the paper, while in Section 5 we discuss conclusions and open problems deriving from our work.

2. Preliminaries and definitions

We assume familiarity with basic concepts of graph drawing and planarity (see, e.g., [12]). Let $G(V, E)$ be a graph and let Γ be a drawing of G in the plane. If all vertices and edge bends of Γ have integer coordinates, then Γ is a *grid drawing* of G, and the *area* of Γ is the area of the minimum bounding box of Γ. We recall that the minimum bounding box of a drawing Γ is the rectangle of minimum area enclosing Γ. If Γ is not on an integer grid, we scale it in order to guarantee the same resolution rule of a grid drawing; namely we expect that the minimum Euclidean distance between any two points on which either vertices or bends of Γ are drawn is at least of one unit. Under this resolution rule, we define the area of the drawing as the area of the minimum bounding box of Γ.

Let $G(V, E)$ be a graph and let $S(V, W)$, $W \subseteq E$, be a spanning subgraph of G. A straight-line drawing Γ of G such that S is crossing-free in Γ (i.e., such that crossings occur only between edges of $E \setminus W$) is called a *straight-line compatible drawing* of $\langle G, S\rangle$. If each edge of $E \setminus W$ has at most k bends in Γ (but still S is drawn straight-line and crossing-free in Γ), Γ is called a *k-bend compatible drawing* of $\langle G, S\rangle$.

If S is a rooted spanning tree of G such that every edge of G not in S connects either vertices at the same level of S or vertices that are on consecutive levels, then we say that S is a *proper level spanning tree* of G.

A *star* is a tree $T(V, E)$ such that all its vertices but one have degree one, that is, $V = \{u, v_1, v_2, \ldots, v_k\}$ and $E = \{(u, v_1), (u, v_2), \ldots, (u, v_k)\}$; any subdivision of T (including T), is a *spider*: vertex u is the *center* of the spider and each path from u to v_i is a *leg* of the spider. A *caterpillar* is a tree such that removing all its leaves (and their incident edges) results in a path, which is called the *spine* of the caterpillar. The one-degree vertices attached to a spine vertex v are called the *leaves* of v.

In the remainder of the paper we implicitly assume that G is always a connected graph (if the graph is not connected, our results apply for any connected component).