Computational Geometry 48 (2015) 622–633

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Bichromatic compatible matchings

Greg Aloupis [a,1], Luis Barba [a,b,*], Stefan Langerman [a,2], Diane L. Souvaine [c,3]

[a] Département d'Informatique, Université Libre de Bruxelles, Brussels, Belgium
[b] School of Computer Science, Carleton University, Ottawa, Canada
[c] Department of Computer Science, Tufts University, Medford, MA, United States

ARTICLE INFO

Article history:
Available online 26 August 2014

Keywords:
Perfect matchings
Bichromatic point set
Compatible matchings
Transformation graph
Reconfiguration problem

ABSTRACT

For a set R of n red points and a set B of n blue points, a *BR-matching* is a non-crossing geometric perfect matching where each segment has one endpoint in B and one in R. Two *BR*-matchings are compatible if their union is also non-crossing. We prove that, for any two distinct *BR*-matchings M and M', there exists a sequence of *BR*-matchings $M = M_1, \dots, M_k = M'$ such that M_{i-1} is compatible with M_i. This implies the connectivity of the *compatible bichromatic matching graph* containing one node for each *BR*-matching and an edge joining each pair of compatible *BR*-matchings, thereby answering the open problem posed by Aichholzer et al. in [1].

1. Introduction

A planar straight line graph (PSLG) is a geometric graph in which the vertices are points embedded in the plane and the edges are non-crossing line segments. There are many special types of PSLGs of which we name a few. A triangulation is a PSLG to which no more edges may be added between existing vertices. A *geometric matching* of a given point set P is a 1-regular PSLG consisting of pairwise disjoint line segments in the plane joining points of P. A geometric matching is *perfect* if every point in P belongs to exactly one segment.

Two branches of study on PSLGs include those of geometric augmentation and geometric reconfiguration. A typical augmentation problem on a PSLG $G = (V, E)$ asks for a set of new edges E' such that the graph $(V, E \cup E')$ retains or gains some desired properties (see survey by Hurtado and Tóth [2]).

A typical reconfiguration problem on a pair of PSLGs G and G' sharing some property asks for a sequence of PSLGs $G = G_0, \dots, G_k = G'$ where each successive pair of PSLGs G_{i-1}, G_i jointly satisfies some geometric constraints. In some situations, a bound on the value of k is desired as well [3–9].

One such solved problem is that of reconfiguring triangulations: given two triangulations T and T', one can compute a sequence of triangulations $T = T_0, \dots, T_k = T'$ on the same point set such that T_{i-1} can be reconfigured to T_i by flipping one edge. Furthermore, bounds on the value of k are known: $O(n^2)$ edge flips are always sufficient [8] and $\Omega(n^2)$ edge flips are sometimes necessary [7].

* Corresponding author.
E-mail addresses: aloupis.greg@gmail.com (G. Aloupis), lbarbafl@ulb.ac.be (L. Barba), slanger@ulb.ac.be (S. Langerman), dls@cs.tufts.edu (D.L. Souvaine).
1 Chargé de recherches du F.R.S.-FNRS.
2 Directeur de recherches du F.R.S.-FNRS.
3 Supported by NSF Grant #CCF-0830734.

http://dx.doi.org/10.1016/j.comgeo.2014.08.009
0925-7721/

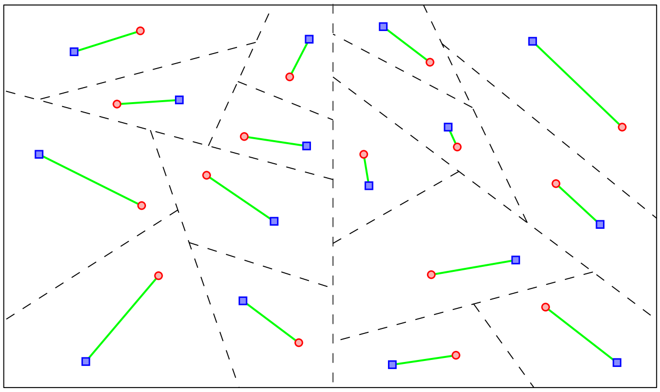

Fig. 1. A ham-sandwich matching obtained by recursively applying ham-sandwich cuts.

Two PSLGs (not necessarily disjoint) on the same vertex set are *compatible* if their union is planar. Compatible geometric matchings have been the object of study in both augmentation and reconfiguration problems. For example, the *Disjoint Compatible Matching Conjecture* [5] was recently solved in the affirmative [10]: every perfect planar matching M of $2n$ segments on $4n$ points can be augmented by $2n$ additional segments to form a PSLG that is the union of simple polygons.

Let M and M' be two perfect planar matchings of a given point set. The reconfiguration problem asks for a *compatible sequence* of perfect matchings $M = M_0, \ldots, M_k = M'$ such that M_{i-1} is compatible with M_i for all $i \in \{1, \ldots, k\}$. Aichholzer et al. [5] proved that there is always a compatible sequence of $O(\log n)$ matchings that reconfigures any given matching into a canonical matching. Thus, the *compatible matching graph*, that has one node for each perfect planar matching and an edge between any two compatible matchings, is connected with diameter $O(\log n)$. Razen [9] proved that the distance between two nodes in this graph is sometimes $\Omega(\log n/\log\log n)$.

A natural question to extend this research is to ask what happens with bichromatic point sets in which the segments must join points from different colors. Let $P = B \cup R$ be a set of points in the plane in general position where $|R| = |B| = n$. A straight-line segment with one endpoint in B and one in R is called a *bichromatic segment*. A perfect planar matching of P where every segment is bichromatic is called a *BR-matching*. Sharir and Welzl [11] proved that the number of *BR*-matchings of P is at most $O(7.61^n)$. Hurtado et al. [12] showed that any *BR*-matching can be augmented to a crossing-free bichromatic spanning tree in $O(n \log n)$ time. Aichholzer et al. [1] proved that for any *BR*-matching M of P, there are at least $\lceil \frac{n-1}{2} \rceil$ bichromatic segments spanned by P that are compatible with M. Furthermore, there are *BR*-matchings with at most $3n/4$ compatible bichromatic segments.

At least one *BR*-matching can always be produced by recursively applying *ham-sandwich cuts*; see Fig. 1 for an illustration. A *BR*-matching produced in this way is called a *ham-sandwich matching*. Notice that the general position assumption is sometimes necessary to guarantee the existence of a *BR*-matching. However, not all *BR*-matchings can be produced using ham-sandwich cuts. Furthermore, some point sets admit only one *BR*-matching, which must be a ham-sandwich matching.

Two *BR*-matchings M and M' are *connected* if there is a sequence of *BR*-matchings $M = M_0, \ldots, M_k = M'$, such that M_{i-1} is compatible with M_i, for $1 \le i \le k$. An open problem posed by Aichholzer et al. [1] was to prove that all *BR*-matchings of a given point set are connected.[4] We answer this in the affirmative by using a ham-sandwich matching H as a canonical form. Consider the first ham-sandwich cut line ℓ used to construct H. We show how to reconfigure any given *BR*-matching via a compatible sequence so that the last matching in the sequence contains no segment crossing ℓ. We use this result recursively, on every ham-sandwich cut used to generate H, to show that any given *BR*-matching is connected with H.

2. Ham-sandwich matchings

In this paper, a *ham-sandwich cut* of P is a line passing through no point of P and containing exactly $\lfloor \frac{n}{2} \rfloor$ blue and $\lfloor \frac{n}{2} \rfloor$ red points to one side. Notice that if n is even, then this matches the *classical* definition of ham-sandwich cuts (see Chapter 3 of [13]). However, when n is odd, a ham-sandwich cut ℓ according to the classical definition will go through a red and a blue point of P. In this case, we obtain a ham-sandwich cut according to our definition by slightly moving ℓ away from these two points without changing its slope and without reaching another point of P. By the general position assumption, this is always possible.

Recall that P admits at least one ham-sandwich matching resulting from recursively applying ham-sandwich cuts. Moreover, note that P may admit several ham-sandwich matchings.

Let M be a *BR*-matching of P. In this section we prove that M is connected with a ham-sandwich matching H of P. Consider a ham-sandwich cut ℓ used to construct H. The idea of the proof is to show the existence of a *BR*-matching M',

[4] This problem was also posed during the EuroGIGA meeting that took place after EuroCG 2012.