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Higher-order triangular-distance Delaunay graphs: Graph-theoretical properties ☆

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ABSTRACT

We consider an extension of the triangular-distance Delaunay graphs (TD-Delaunay) on a set P of points in general position in the plane. In TD-Delaunay, the convex distance is defined by a fixed-oriented equilateral triangle $\triangledown$, and there is an edge between two points in P if and only if there is an empty homothet of $\triangledown$ having the two points on its boundary. We consider higher-order triangular-distance Delaunay graphs, namely k-TD, which contains an edge between two points if the interior of the smallest homothet of $\triangledown$ having the two points on its boundary contains at most k points of P. We consider the connectivity, Hamiltonicity and perfect-matching admissibility of k-TD. Finally we consider the problem of blocking the edges of k-TD.

1. Introduction

The *triangular-distance Delaunay graph* of a point set P in the plane, TD-Delaunay for short, was introduced by Chew [12]. A TD-Delaunay is a graph whose convex distance function is defined by a fixed-oriented equilateral triangle. Let $\triangledown$ be a downward equilateral triangle whose barycenter is the origin and one of whose vertices is on the negative y-axis. A *homothet* of $\triangledown$ is obtained by scaling $\triangledown$ with respect to the origin by some factor $\mu \geq 0$, followed by a translation to a point b in the plane: $b + \mu\triangledown = \{b + \mu a : a \in \triangledown\}$. In the TD-Delaunay graph of P, there is a straight-line edge between two points p and q if and only if there exists a homothet of $\triangledown$ having p and q on its boundary and whose interior does not contain any point of P. In other words, (p, q) is an edge of TD-Delaunay graph if and only if there exists an empty downward equilateral triangle having p and q on its boundary. In this case, we say that the edge (p, q) has the *empty triangle property*.

We say that P is in general position if the line passing through any two points from P does not make angles 0°, 60°, and 120° with horizontal. In this paper we consider point sets in general position and our results assume this pre-condition. If P is in general position, then the TD-Delaunay graph is a planar graph, see [7]. We define $t(p, q)$ as the smallest homothet of $\triangledown$ having p and q on its boundary. See Fig. 1(a). Note that $t(p, q)$ has one of p and q at a vertex, and the other one on the opposite side. Thus,

Observation 1. *Each side of $t(p, q)$ contains either p or q.*

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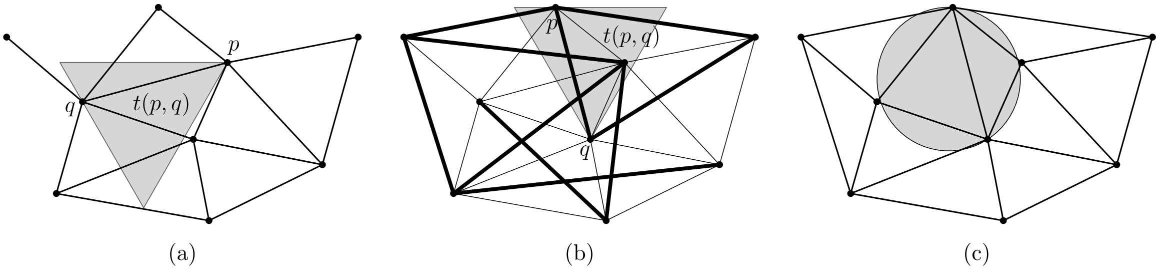

Fig. 1. (a) Triangular-distance Delaunay graph (0-TD), (b) 1-TD graph, the light edges belong to 0-TD as well, and (c) Delaunay triangulation.

A graph G is *connected* if there is a path between any pair of vertices in G. Moreover, G is *k-connected* if there does not exist a set of at most $k-1$ vertices whose removal disconnects G. In case $k=2$, G is called *biconnected*. In other words a graph G is biconnected iff there is a simple cycle between any pair of its vertices. A *matching* in G is a set of edges in G without common vertices. A *perfect matching* is a matching which matches all the vertices of G. A *Hamiltonian cycle* in G is a cycle (i.e., closed loop) through G that visits each vertex of G exactly once. For $H \subseteq G$ we denote the *bottleneck* of H, i.e., the length of the longest edge in H, by $\lambda(H)$.

Let $K_n(P)$ be a complete edge-weighted geometric graph on a point set P which contains a straight-line edge between any pair of points in P. For an edge (p,q) in $K_n(P)$ let $w(p,q)$ denote the weight of (p,q). A *bottleneck matching* (resp. *bottleneck Hamiltonian cycle*) in P is defined to be a perfect matching (resp. Hamiltonian cycle) in $K_n(P)$, in which the weight of the maximum-weight edge is minimized. A *bottleneck biconnected spanning subgraph* of P is a spanning subgraph, $G(P)$, of $K_n(P)$ which is biconnected and the weight of the longest edge in $G(P)$ is minimized.

A tight lower bound on the size of a maximum matching in a TD-Delaunay graph, i.e. 0-TD, is presented in [4]. In this paper we study higher-order TD-Delaunay graphs. The *order-k TD-Delaunay graph* of a point set P, denoted by k-TD, is a geometric graph which has an edge (p,q) iff the interior of $t(p,q)$ contains at most k points of P; see Fig. 1(b). The standard TD-Delaunay graph corresponds to 0-TD. We consider graph-theoretic properties of higher-order TD-Delaunay graphs, such as connectivity, Hamiltonicity, and perfect-matching admissibility. We also consider the problem of blocking TD-Delaunay graphs.

1.1. Previous work

A *Delaunay triangulation* (DT) of P (which does not have any four co-circular points) is a graph whose distance function is defined by a fixed circle $\bigcirc$ centered at the origin. DT has an edge between two points p and q iff there exists a homothet of $\bigcirc$ having p and q on its boundary and whose interior does not contain any point of P; see Fig. 1(c). In this case the edge (p,q) is said to have the *empty circle property*. The *order-k Delaunay Graph* on P, denoted by k-DG, is defined to have an edge (p,q) iff there exists a homothet of $\bigcirc$ having p and q on its boundary and whose interior contains at most k points of P. The standard Delaunay triangulation corresponds to 0-DG.

For each pair of points $p,q \in P$ let $D[p,q]$ be the closed disk having pq as diameter. A *Gabriel Graph* on P is a geometric graph which has an edge between two points p and q iff $D[p,q]$ does not contain any point of $P \setminus \{p,q\}$. The *order-k Gabriel Graph* on P, denoted by k-GG, is defined to have an edge (p,q) iff $D[p,q]$ contains at most k points of $P \setminus \{p,q\}$.

For each pair of points $p,q \in P$, let $L(p,q)$ be the intersection of the two open disks with radius $|pq|$ centered at p and q, where $|pq|$ is the Euclidean distance between p and q. A *Relative Neighborhood Graph* on P is a geometric graph which has an edge between two points p and q iff $L(p,q)$ does not contain any point of P. The *order-k Relative Neighborhood Graph* on P, denoted by k-RNG, is defined to have an edge (p,q) iff $L(p,q)$ contains at most k points of P. It is obvious that for a fixed point set, k-RNG is a subgraph of k-GG, and k-GG is a subgraph of k-DG.

The problem of determining whether an order-k geometric graph always has a (bottleneck) perfect matching or a (bottleneck) Hamiltonian cycle is of interest. In order to show the importance of this problem we provide the following example. Gabow and Tarjan [15] showed that a bottleneck matching of maximum cardinality in a graph can be computed in $O(m \cdot (n \log n)^{0.5})$ time, where m is the number of edges in the graph. Using their algorithm, a bottleneck perfect matching of a point set can be computed in $O(n^2 \cdot (n \log n)^{0.5})$ time; note that the complete graph on n points has $\Theta(n^2)$ edges. Chang et al. [11] showed that a bottleneck perfect matching of a point set is contained in 16-DG; this graph has $\Theta(n)$ edges and can be computed in $O(n \log n)$ time. Thus, by running the algorithm of Gabow and Tarjan on 16-DG, a bottleneck perfect matching of a point set can be computed in $O(n \cdot (n \log n)^{0.5})$ time.

If for each edge (p,q) in $K_n(P)$, $w(p,q)$ is equal the Euclidean distance between p and q, then Chang et al. [9–11] proved that a bottleneck biconnected spanning graph, bottleneck perfect matching, and bottleneck Hamiltonian cycle of P are contained in 1-RNG, 16-RNG, 19-RNG, respectively. This implies that 16-RNG has a perfect matching and 19-RNG is Hamiltonian. Since k-RNG is a subgraph of k-GG, the same results hold for 16-GG and 19-GG. It is known that k-GG is $(k+1)$-connected [8] and 10-GG (and hence 10-DG) is Hamiltonian [16]. Dillencourt showed that every Delaunay triangulation (0-DG) admits a perfect matching [14] but it can fail to be Hamiltonian [13].