Computational Geometry 48 (2015) 459–478

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Disconnectivity and relative positions in simultaneous embeddings ☆,☆☆

Thomas Bläsius *, Ignaz Rutter *

Karlsruhe Institute of Technology (KIT), Germany

ARTICLE INFO

Article history:
Received 12 April 2013
Accepted 29 January 2015
Available online 7 February 2015

Keywords:
Planarity
Simultaneous embedding
Algorithm
Embedding representation

ABSTRACT

For two planar graphs $G^{\textcircled{1}} = (V^{\textcircled{1}}, E^{\textcircled{1}})$ and $G^{\textcircled{2}} = (V^{\textcircled{2}}, E^{\textcircled{2}})$ sharing a common subgraph $G = G^{\textcircled{1}} \cap G^{\textcircled{2}}$ the problem Simultaneous Embedding with Fixed Edges (SEFE) asks whether they admit planar drawings such that the common graph is drawn the same. Previous algorithms only work for cases where G is connected, and hence do not need to handle relative positions of connected components. We consider the problem where G, $G^{\textcircled{1}}$ and $G^{\textcircled{2}}$ are not necessarily connected.

First, we show that a general instance of SEFE can be reduced in linear time to an equivalent instance where $V^{\textcircled{1}} = V^{\textcircled{2}}$ and $G^{\textcircled{1}}$ and $G^{\textcircled{2}}$ are connected. Second, for the case where G consists of disjoint cycles, we introduce the *CC-tree* which represents all embeddings of G that extend to planar embeddings of $G^{\textcircled{1}}$. We show that CC-trees can be computed in linear time, and that their intersection is again a CC-tree. This yields a linear-time algorithm for SEFE if all k input graphs (possibly $k > 2$) pairwise share the same set of disjoint cycles. These results, including the CC-tree, extend to the case where G consists of arbitrary connected components, each with a fixed planar embedding on the sphere. Then the running time is $O(n^2)$.

1. Introduction

To enable a human reader to compare different relational datasets on a common set of objects it is important to visualize the corresponding graphs in such a way that the common parts of the different datasets are drawn as similarly as possible. An example is a dynamic graph that changes over time. Then the change between two points in time can be easily grasped with the help of a visualization showing the parts that did not change in the same way for both graphs. This leads to the fundamental theoretical problem Simultaneous Embedding with Fixed Edges (or SEFE for short), asking for two graphs $G^{\textcircled{1}} = (V^{\textcircled{1}}, E^{\textcircled{1}})$ and $G^{\textcircled{2}} = (V^{\textcircled{2}}, E^{\textcircled{2}})$ with the common graph $G = (V, E) = (V^{\textcircled{1}} \cap V^{\textcircled{2}}, E^{\textcircled{1}} \cap E^{\textcircled{2}})$, whether there are planar drawings of $G^{\textcircled{1}}$ and $G^{\textcircled{2}}$ such that the common graph G is drawn the same in both.

The problem SEFE and its variants, such as Simultaneous Geometric Embedding, where one insists on a simultaneous straight-line drawing, have been studied intensively in the past years; see the recent survey [1] for an overview. Some of

☆ Part of this work was done within GRADR – EUROGIGA project no. 10-EuroGIGA-OP-003.

☆☆ A preliminary version of this work has been published as T. Bläsius and I. Rutter: Disconnectivity and Relative Positions in Simultaneous Embeddings, *Proceedings of the 20th International Symposium on Graph Drawing (GD'12)*, pages 31–42, LNCS, Springer, 2012.

* Corresponding author.
E-mail addresses: blaesius@kit.edu (T. Bläsius), rutter@kit.edu (I. Rutter).

http://dx.doi.org/10.1016/j.comgeo.2015.02.002
0925-7721/

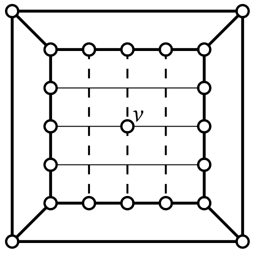

Fig. 1. The bold edges belong to both graphs, the dashed and thin edges are exclusive edges.

the results show, for certain graph classes, that they always admit simultaneous embeddings or that there exist negative instances of SEFE whose input graphs belong to these classes. As there are planar graphs that cannot be embedded simultaneously, the question of deciding whether given graphs admit a SEFE is of high interest. Gassner et al. [2] show that it is $\mathcal{NP}$-complete to decide SEFE for three or more graphs. For two graphs the complexity status is still open. However, there are several approaches yielding efficient algorithms for special cases. Fowler et al. show how to solve SEFE efficiently, if $G^{①}$ and G have at most two and one cycles, respectively [3]. Fowler et al. characterize the class of common graphs that always admit a SEFE [4]. Angelini et al. [5] show that if one of the input graphs has a fixed planar embedding, then SEFE can be solved in linear time. Haeupler et al. solve SEFE in linear time for the case that the common graph is biconnected [6]. Angelini et al. obtain the same result with a completely different approach [7]. They additionally solve the case where the common graph is a star and, moreover, show the equivalence of the case where the common graph is connected to the case where the common graph is a tree and relate it to a constrained book embedding problem. The currently least restrictive result (in terms of connectivity) by Bläsius and Rutter [8] shows that SEFE can be solved in polynomial time for the case that both graphs are biconnected and the common graph is connected.

The algorithms testing SEFE have in common that they use the result by Jünger and Schulz [9] stating that the question of finding a simultaneous embedding for two graphs is equivalent to the problem of finding planar embeddings of $G^{①}$ and $G^{②}$ such that they induce the same embedding on G. Moreover, they have in common that they all assume that the common graph is connected, implying that it is sufficient to enforce the common edges incident to each vertex to have the same circular ordering in both embeddings. Especially in the result by Bläsius and Rutter [8] this is heavily used, as they explicitly consider only orders of edges around vertices using PQ-trees. However, if the common graph is not required to be connected, we additionally have to care about the relative positions of connected components to one another, which introduces an additional difficulty. Note that the case where the common graph is disconnected cannot be reduced to the case where it is connected by inserting additional edges. Fig. 1 shows an instance that admits a simultaneous embedding, which is no longer true if the isolated vertex v is connected to the remaining graph. Other approaches to solve the SEFE problem have only appeared recently. Schaefer [10] characterizes, for certain classes of SEFE instances, the pairs of graphs that admit a SEFE via the independent odd crossing number. Among others, this gives a polynomial-time algorithm for SEFE when the common graph has maximum degree 3 and is not necessarily connected.

In this work we tackle the SEFE problem from the opposite direction than the so far known results, by assuming that the circular order of edges around vertices in G is already fixed and we only have to ensure that the embeddings chosen for the input graphs are *compatible* in the sense that they induce the same relative positions on G. Initially, we assume that the graph G consists of a set of disjoint cycles, each of them having a unique planar embedding. We present a novel data structure, the *CC-tree*, which represents all embeddings of a set of disjoint cycles that can be induced by an embedding of a graph containing them as a subgraph. We moreover show that two such CC-trees can be intersected, again yielding a CC-tree. Thus, for the case that $G^{①}$ and $G^{②}$ have the common graph G consisting of a set of disjoint cycles, the intersection of the CC-trees corresponding to $G^{①}$ and $G^{②}$ represents all simultaneous embeddings. We show that CC-trees can be computed and intersected in linear time, yielding a linear-time algorithm to solve SEFE for the case that the common graph consists of disjoint cycles. Note that this obviously also yields a linear-time algorithm to solve SEFE for more than two graphs if they all share the same common graph consisting of a set of disjoint cycles. We show that these results can be further extended to the case where the common graph may contain arbitrary connected components, each of them with a prescribed planar embedding. However, in this case the corresponding data structure, called $\mathrm{CC}^{\oplus}$-tree, may have quadratic size. These results show that the choice of relative positions of several connected components does not solely make the problem SEFE hard to solve.

Note that these results have an interesting application concerning the problem Partially Embedded Planarity. The input of Partially Embedded Planarity is a planar graph G together with a fixed embedding for a subgraph H (including fixed relative positions). It asks whether G admits a planar embedding extending the embedding of H. Angelini et al. [5] introduced this problem and solve it in linear time. The $\mathrm{CC}^{\oplus}$-tree can be used to solve Partially Embedded Planarity in quadratic time as it represents all possible relative positions of the connected components in H to one another that can be induced by an embedding of G. It is then easy to test whether the prespecified relative positions can be achieved. In fact, this solves the slightly more general case of Partially Embedded Planarity where not all relative positions have to be fixed.