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On the area requirements of Euclidean minimum spanning trees ☆

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ABSTRACT

In their seminal paper on Euclidean minimum spanning trees, Monma and Suri (1992) proved that any tree of maximum degree 5 admits a planar embedding as a Euclidean minimum spanning tree. Their algorithm constructs embeddings with exponential area; however, the authors conjectured that there exist n-vertex trees of maximum degree 5 that require $c^n \times c^n$ area to be embedded as Euclidean minimum spanning trees, for some constant $c > 1$. In this paper, we prove the first exponential lower bound on the area requirements for embedding trees as Euclidean minimum spanning trees.

1. Introduction

A *Euclidean minimum spanning tree* (MST) of a set P of points in the plane is a tree with a vertex in each point of P and with minimum total edge length. Euclidean minimum spanning trees have several applications in computer science and hence they have been deeply investigated from a theoretical point of view. To cite a few major results, optimal $\Theta(n \log n)$-time algorithms are known to compute an MST of any set of points and it is $\mathcal{NP}$-hard to compute an MST with maximum degree bounded by 2, 3, or 4 [6,8,15], while polynomial-time algorithms exist [1,3,10,13] to compute spanning trees with maximum degree bounded by 2, 3, or 4 and total edge length within a constant factor from the optimal one.

An *MST embedding* of a tree T is a plane embedding of T such that the MST of the points where the vertices of T are drawn coincides with T. In this paper we consider the problem of constructing MST embeddings of trees of maximum degree 5. Several results are known related to such a problem. No tree having a vertex of degree at least 7 admits an MST embedding. Further, deciding whether a tree with degree 6 admits an MST embedding is $\mathcal{NP}$-hard [5]. However, restricting the attention to trees of degree 5 is not a limitation since: (i) every planar point set has an MST with maximum degree 5 [14], and (ii) every tree of maximum degree 5 admits an MST embedding in the plane [14].

MST embeddings have also been considered as a particular type of proximity drawings (see, e.g., [2,11]), where adjacent vertices have to be placed "close" to each other. From the various applications, different measures for closeness have been introduced and evaluated, and different graph classes that can be realized under the specific closeness measure have been studied together with the area/volume that is needed for their realization. Prominent examples are Delaunay triangulations, Gabriel drawings, nearest-neighbor drawings, β-drawings, and many more. The MST constraints can be formulated as closeness conditions with respect to pairs of vertices, either adjacent or non-adjacent.

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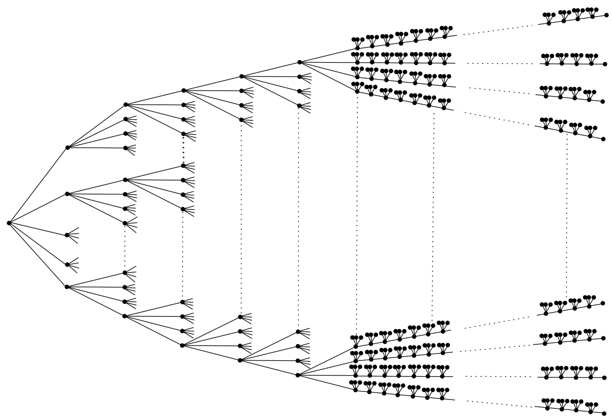

Fig. 1. A tree T^* requiring $2^{\Omega(n)}$ area in any MST embedding.

Monma and Suri's proof [14] that every tree of maximum degree 5 admits an MST embedding in the plane is a strong combinatorial result; on the other hand, their algorithm for constructing MST embeddings seems to be useless in practice, since the constructed embeddings have $2^{\Theta(k^2)}$ area for trees of height k (hence, in the worst case the area requirement of such drawings is $2^{\Theta(n^2)}$). However, Monma and Suri conjectured that there exist trees of maximum degree 5 that require $c^n \times c^n$ area in *any* MST embedding, for some constant $c > 1$. The problem of determining whether or not the area upper bound for MST embeddings of trees can be improved to polynomial is reported also in [4,5,9,12]. Recently, MST embeddings in polynomial area have been proven to exist for trees with maximum degree 4 [7].

In this paper, we prove that there exist n-vertex trees of maximum degree 5 requiring $2^{\Omega(n)}$ area in any MST embedding. Our lower bound is achieved by considering an n-vertex tree T^*, shown in Fig. 1, composed of a degree-5 *complete tree* T_c with a constant number of vertices and of a set of degree-5 *caterpillars*, each one attached to a distinct leaf of T_c. The argument goes in two steps: For the first step, we walk down the tree T_c, starting from the root. The route is chosen so that the angles adjacent to the edges are narrowing at each step. The key observation here is a lemma relating the size of two consecutive angles adjacent to an edge. At the leaves of the complete tree T_c, where the caterpillars start, the angles incident to an end-vertex of the backbone of at least one of the caterpillars must be very small, that is, between 60° and 61°. Using this as a starting point, in the second step we prove that each angle incident to a vertex of the caterpillar is either very small, that is, between 60° and 61°, or is very large, that is, between 89.5° and 90.5°. As a consequence, we show that when walking down along the backbone of the caterpillar, the lengths of the edges decrease exponentially along the caterpillar. Since the backbone has a linear number of edges, we obtain the claimed area bound.

The paper is organized as follows. In Section 2 we give definitions and preliminaries; in Section 3 we give some geometric lemmata; in Section 4 we argue about angles and edge lengths of the MST embeddings of T^*; in Section 5 we prove the area lower bound; in Section 6, we conclude with some remarks and a conjecture.

2. Preliminaries

A *rooted tree* is a tree with one distinguished vertex, called *root*. The *depth* of a vertex in a rooted tree is its distance from the root, that is, the number of edges in the path from the root to the vertex. The *height* of a rooted tree is the maximum depth of one of its vertices. A *complete tree* is a rooted tree such that every path from the root to a leaf has the same number of vertices and every vertex has the same degree. A *caterpillar* is a tree such that removing the leaves yields a path, called the *backbone* of the caterpillar.

A *minimum spanning tree* MST of a planar point set P is a tree spanning P with minimum total edge length. Given a tree T, an *MST embedding* of T is a straight-line drawing of T such that the MST of the points where the vertices of T are drawn is isomorphic to T. The *area* of an MST embedding is the area of the smallest rectangle enclosing such an embedding. The concept of area of an MST embedding only makes sense once a *resolution rule* is fixed, i.e., a rule that does not allow vertices to be arbitrarily close (*vertex resolution rule*), or edges to be arbitrarily short (*edge resolution rule*). Without any of such rules, one could just construct MST embeddings with arbitrarily small area. In the following we will hence suppose that any two vertices have distance at least one unit. Then, in order to prove that an n-vertex tree T requires $\Omega(f(n))$ area in any MST embedding, it suffices to prove that the ratio between the longest and the shortest edge of any MST embedding is $\Omega(f(n))$, and that both dimensions have at least $\Omega(1)$ size.

Consider any MST embedding of a tree T rooted at a vertex r. The *clockwise path* $Cl(u)$ of a vertex $u \neq r$ of T is the path $v_0, \ldots, v_k$ such that $v_0 = u$, (v_i, v_{i+1}) is the edge following the edge from v_i to its parent in the clockwise order of the edges incident to v_i, for $i = 0, \ldots, k-1$, and v_k is a leaf. The *counterclockwise path* $Ccl(u)$ of a vertex $u \neq r$ of T is defined analogously. Denote by $d(a, b)$ the Euclidean distance between two vertices a and b (or between two points a and b) and by $|e|$ the Euclidean length of an edge e. Further, $k(c, r)$ denotes the circle centered at a point c with radius r.