Computational Geometry 47 (2014) 25–41

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Upward planar drawings on the standing and the rolling cylinders ☆

Franz J. Brandenburg

University of Passau, 94030 Passau, Germany

ARTICLE INFO

Article history:
Received 25 October 2012
Received in revised form 29 May 2013
Accepted 20 June 2013
Available online 2 September 2013
Communicated by D. Wagner

Keywords:
Graph drawing
Upward planar graphs
Polyline drawings
Cylinders

ABSTRACT

We consider directed graphs with an upward planar drawing on the standing and the rolling cylinders. These graphs properly extend the upward planar graphs in the plane. The drawings allow complex curves for the edges with many zig-zags and windings around the cylinder.

We show that the drawings can be simplified to polyline drawings with geodesics as straight segments, at most two bends per edge, and vertices and bends on a grid of at most cubic size. On the standing cylinder the edges do not wind and they wind at most once around the rolling cylinder, where single windings can be unavoidable. The simplifications can be computed efficiently in $\mathcal{O}(\tau n^3)$ time, where τ is the cost of computing the point of intersection of a curve and a horizontal line through a vertex.

Moreover, we provide a complete classification of the classes of (regular) upward planar drawable graphs on the plane, the sphere, and the standing and rolling cylinders including their strict and weak variants. Strict is upward in two dimensions and weak allows horizontal lines. For upward planarity the standing and the rolling cylinders coincide in the strict case, rolling dominates standing in the regular case, and there is an incomparability in the weak case. The respective classes of graphs range properly between the upward planar and the quasi-upward planar graphs, and they have an $\mathcal{NP}$-hard recognition problem.

1. Introduction

Graph drawing is mostly concerned with the problem to map a graph in the plane. The vertices are mapped to points and the edges to Jordan curves between the endpoints. The objective is a nice drawing, which shall help the user to better understand the structural relations modeled by the graph. Nice is formalized by aesthetics of the drawing, such as the numbers of crossings and bends and the used area. Aesthetics convert to restrictions if they are absolute, such as planar and straight line. Here we add unidirectional and shall consider upward planar graphs on the plane, the sphere, and the standing and the rolling cylinders.

The task of drawing a graph consists of two phases: placement of the vertices and routing of the edges. The standard is polylines for the edges with straight segments and bends [17,34]. So the curves have a discrete description and the drawing is completely specified by the coordinates for the vertices and bends. In the straight-line approaches the placement and routing phases are merged. More complex curves are used in freehand drawings and in a postprocessing phase if bends of polylines are smoothed by splines [21] or spiral segments [5] or in Lombardi drawings with circular arcs [24].

☆ Supported in part by the Deutsche Forschungsgemeinschaft (DFG), grant Br835/15-2.
E-mail address: brandenb@informatik.uni-passau.de.

0925-7721/$ – see front matter
http://dx.doi.org/10.1016/j.comgeo.2013.08.003

The full generality of Jordan curves is used in the definition of planar graphs and planar drawings. A drawing of a graph is planar if it is one-to-one on the vertices and the curves are disjoint except at common endpoints. What does this really mean? How complex are the curves? What do we know if we are given a planar drawing of a graph as a witness for planarity (or any other property which depends on a drawing)? Jordan curves are continuous but not necessarily computable. The curves may be highly complex with many turns and many windings around a cylinder. There may even be infinitely many zig-zags, e.g., driven by the function $f(t) = t \sin \frac{1}{t}$ with $t \to 0$ if we accept this as a description. Infinitely many windings with an ultimate convergence towards an endpoint are excluded by the continuity of a Jordan curve. However, the number of windings can be arbitrarily large. For example, a curve may form a spiral with successive windings around the cylinder at distance 2^{-i} for $i = 1, \ldots, N$, and N is, e.g., the value of Ackermann's function $A(n, m)$ with n and m as the numbers of vertices and edges. For effective computations the curves must have a finite description, which is given by a complete listing, a program or a Turing machine. However, such a description may be very long and exceed the size of the graph by any (super-exponential) bound. Such a complexity would be out of scope of graph drawing and against common standards.

To cope with the complexity of the description of Jordan curves we suppose that it takes time τ to compute the coordinates of the intersection of an edge and a horizontal line through a vertex. τ may be huge. In the previous example with windings described by Ackermann's function one must evaluate the function. However, τ is at most $\mathcal{O}(1)$ in the resulting drawings. On the standing cylinder and the sphere the polyline drawing can be computed in $\mathcal{O}(\tau n^2)$ time, and it takes $\mathcal{O}(\tau n^3)$ time on the rolling cylinder and graphs of size n. This is a significant reduction of the description complexity of upward planar drawings. The time bounds stem from the number of observable points where the curves are evaluated. This seems easier than tracing the edges, particularly, if they zig-zag and frequently wind around the cylinder. So we bridge the gap between drawings with arbitrary Jordan curves and polyline drawings, and we capture Jordan curves computationally using a value τ for the computation of an intersection of curves in a neighborhood of vertices.

The accepted standard is polyline drawings. The common graph drawing algorithms fulfill this standard and draw undirected graphs straight-line using force or energy based methods and draw directed graphs with few (at most two) bends per edge using the hierarchical approach [17,34]. Also every planar graph has a planar polyline drawing as shown in [40]. The proof is based on the existence of open discs, and does not provide an effective construction. In addition, it is well-known that planar graphs admit straight-line drawings. This was first proved by Steinitz and Rademacher [43], Wagner [46] and Stein [42] using the one-to-one correspondence between (triconnected) planar graphs and the mesh of convex polyhedra. An alternative proof by Fáry [26] uses induction and is based on the fact that the outer face of a triangulated planar graph can be drawn as a straight-line triangle. These investigations were finalized by de Fraysseix, Pach and Pollack [28] and by Schnyder [41] who showed that every planar graph has a straight-line grid drawing of $\mathcal{O}(n^2)$ area, which can be computed in $\mathcal{O}(n)$ time.

Let's consider direction. Directed graphs are commonly drawn as hierarchies using the approach of Sugiyama et al. [44]. For acyclic graphs this drawing style transforms the edge direction into a geometric direction: all edges point upward. Otherwise, the approach proposes heuristics to redirect some edges and break all cycles. Alternatively, one may follow the recurrent hierarchy approach suggested by Sugiyama et al. and elaborated by Bachmaier et al. [7]. It identifies the upper and lower sides of a page and draws graphs on the surface of a rolling cylinder such that all edges are upward. The new paradigm is: go with the flow and display cycles unidirectionally.

In this work we combine direction and planarity. We insist on both criteria and do not allow exceptions. In the plane this leads to upward planar drawings. A graph is *upward planar* or an *UP graph* if it can be embedded in the plane such that the curves of the edges are monotonically increasing in y-direction and there are no edge crossings. This is the regular case. Let **UP** denote the respective class of graphs. UP graphs were intensively studied since the late 1980's. They were characterized as the spanning subgraphs of planar st-graphs, which are acyclic planar directed graphs with a single source s and a single sink t and an edge from s to t [19,35]. Concerning the curve complexity upward planar graphs admit straight-line upward drawings which may require an area of exponential size [20], or upward polyline drawings on quadratic area with at most two bends per edge [19]. The recognition problem is $\mathcal{NP}$-hard [29], in general, whereas there are efficient polynomial time algorithms if the graphs are given with an embedding or are triconnected [13] or have a single source [14].

Upward planarity has been generalized to specific surfaces of solids in $\mathcal{R}^3$ and in particular the sphere, the standing cylinder and the torus, see also [23,25,27,31,38,39]. A graph G is *spherical upward planar* or an *SUP graph* if it has an upward planar drawing on the sphere or equivalently on the standing cylinder. G is *rolling upward planar* or an *RUP graph* if it has an upward planar drawing on the rolling cylinder. Let **SUP** and **RUP**, denote the respective classes of graphs. They are central for this work, which extends and improves [15].

Note that the surfaces of genus zero no longer coincide for upward planarity: the plane is weaker than the sphere or the standing cylinder and these are weaker than the rolling cylinder. The graph from Fig. 1, called the 'house', is not upward planar [17,35], but is can be drawn upward planar on the cylinders routing some edges over the backside, as Fig. 2 demonstrates. The 'triple-house' graph from Fig. 10 illustrates the differences between UP, SUP and RUP graphs by the presence of zero to three of the curved edges. On the other hand, for every directed planar graph G there is a surface $\mathcal{S}$ of genus zero such that G has an upward drawing on $\mathcal{S}$ [25]. $\mathcal{S}$ is obtained from a 2D drawing of the undirected graph and the third dimension is used for the upward direction and is computed by topological sorting.

The properties of graphs with upward planar drawings on the sphere and the standing cylinder parallel those from the plane. The distinction to the plane is an st-edge from the source to the sink, which prohibits routing edges over the