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The visible perimeter of an arrangement of disks ☆

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Given a collection of n opaque unit disks in the plane, we want to find a stacking order for them that maximizes their *visible perimeter*, the total length of all pieces of their boundaries visible from above. We prove that if the centers of the disks form a *dense* point set, *i.e.*, the ratio of their maximum to their minimum distance is $O(n^{1/2})$, then there is a stacking order for which the visible perimeter is $\Omega(n^{2/3})$. We also show that this bound cannot be improved in the case of a sufficiently small $n^{1/2} \times n^{1/2}$ uniform grid. On the other hand, if the set of centers is dense and the maximum distance between them is small, then the visible perimeter is $O(n^{3/4})$ with respect to any stacking order. This latter bound cannot be improved either.

Finally, we address the case where no more than c disks can have a point in common.

These results partially answer some questions of Cabello, Haverkort, van Kreveld, and Speckmann.

1. Introduction

In cartography and data visualization, one often has to place similar copies of a symbol, typically an opaque disk, on a map or a figure at given locations [3,6]. The size of the symbol is sometimes proportional to the quantitative data associated with the location. On a cluttered map, it is difficult to identify the symbols. Therefore, it has been investigated in several studies how to minimize the amount of overlap [7,10].

In the present note, we follow the approach of Cabello, Haverkort, van Kreveld, and Speckmann [2]. We assume that the symbols used are opaque circular disks of the same size. Given a collection $\mathcal{D}$ of n distinct *unit* disks in the (x, y)-plane, a *stacking order* is a one-to-one assignment $f : \mathcal{D} \to \{1, 2, \dots, n\}$. We consider the integer $f(D)$ to be the z-coordinate of the disk $D \in \mathcal{D}$. The *map* corresponding to this stacking order is the 2-dimensional view of this arrangement from the point at negative infinity of the z-axis (for notational convenience, we look at the arrangement from below rather than from above.) In particular, for the lowest disk D, we have $f(D) = 1$, and this disk, including its full perimeter, is visible from below. The total length of the boundary pieces of the disks visible from below is the *visible perimeter* of $\mathcal{D}$ with respect to the stacking order f, denoted by visible$(\mathcal{D}, f)$. We are interested in finding a stacking order for which the visible perimeter of $\mathcal{D}$ is as large as possible. See Fig. 1.

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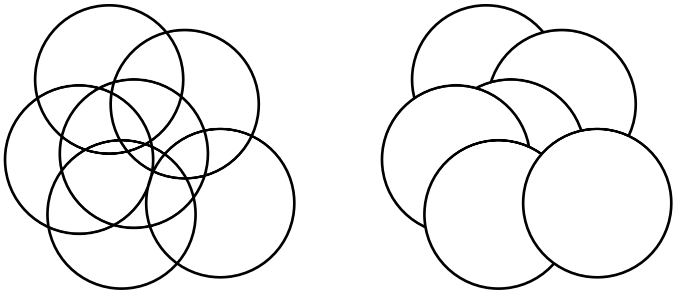

Fig. 1. Left: A collection of unit disks in the plane. Right: A stacking order for them.

There are other situations in which this setting is relevant. Sometimes the vertices of a graph are not represented as points but as circles of a given radius. It may happen that some vertices overlap in the visualization (especially if they have further constraints on their geometric position), and then it becomes important to choose a convenient stacking order that maximizes the visible perimeter.

Given an integer n, we define

$$v(n) = \inf_{|\mathcal{D}|=n} \max_{f} \text{visible}(\mathcal{D}, f), \tag{1}$$

where the maximum is taken over all stacking orders f. We would like to describe the asymptotic behavior of $v(n)$, as n tends to infinity.

Cabello et al. have already noted that $v(n) = \Omega(n^{1/2})$; in other words, every set $\mathcal{D}$ of n disks of unit radii admits a stacking order with respect to which its visible perimeter is $\Omega(n^{1/2})$. Indeed, by a well-known result or Erdős and Szekeres [4], we can select a sequence of $\lceil n^{1/2} \rceil$ disks $D_i \in \mathcal{D}$ $(1 \leqslant i \leqslant \lceil n^{1/2} \rceil)$ such that their centers form a monotone sequence. More precisely, letting x_i and y_i denote the coordinates of the center of D_i, we have $x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots$ and either $y_1 \leqslant y_2 \leqslant y_3 \leqslant \cdots$ or $y_1 \geqslant y_2 \geqslant y_3 \geqslant \cdots$. Then, in any stacking order f such that $f(D_i) = i$ for every i, $1 \leqslant i \leqslant \lceil n^{1/2} \rceil$, a full quarter of the perimeter of each D_i $(1 \leqslant i \leqslant \lceil n^{1/2} \rceil)$ is visible from below. Therefore, the visible perimeter of $\mathcal{D}$ with respect to f satisfies

$$\text{visible}(\mathcal{D}, f) \geqslant \frac{\pi}{2} \lceil n^{1/2} \rceil.$$

At the problem session of *EuroCG*'11 (Morschach, Switzerland), Cabello, Haverkort, van Kreveld, and Speckmann asked whether $v(n) = \Omega(n)$; in other words, does there exist a positive constant c such that every set of n unit disks in the plane admits a stacking order, with respect to which its visible perimeter is at least cn? We answer this question in the negative; *cf.* Theorems 2 and 5 below.

Given a set of points P in the plane, let $\mathcal{D}(P)$ denote the collection of disks of radius 1 centered at the elements of P. For any positive real ε, let εP stand for a similar copy of P, scaled by a factor of ε. For a stacking order f of $\mathcal{D}(P)$ we will study the quantity visible$(\mathcal{D}(\varepsilon P), f)$. (Note the slight abuse of notation: We denote the stacking order of $\mathcal{D}(P)$ and the corresponding stacking order of $\mathcal{D}(\varepsilon P)$ by the same symbol f. The two orders are also identified in Lemmas 1 and 7 and in Theorems 2, 3, and 5.) It is not hard to verify that, as ε gets smaller, the function visible$(\mathcal{D}(\varepsilon P), f)$ decreases. To see this, it is enough to observe, as was also done by Cabello et al. (unpublished), that as we contract the set of centers, the part of the boundary of each unit disk visible from below shrinks. As we will see in Lemma 7, the limit in the following lemma has a simple alternative geometric interpretation.

Lemma 1. *For every point set P in the plane and for every stacking order f of the collection of disks $\mathcal{D}(P)$, we have*

$$\text{visible}\bigl(\mathcal{D}(P), f\bigr) \geqslant \lim_{\varepsilon \to 0} \text{visible}\bigl(\mathcal{D}(\varepsilon P), f\bigr).$$

As in [1,11,12], we consider *C-dense n*-element point sets P, *i.e.*, point sets in which the ratio of the maximum distance between two points to the minimum distance satisfies

$$\frac{\max(|pq|\colon p, q \in P)}{\min(|pq|\colon p, q \in P, p \neq q)} \leqslant Cn^{1/2}.$$

(The above ratio is sometimes called the *spread* of P [5]; thus, we consider point sets with spread at most $Cn^{1/2}$.)

Theorem 2. *For any C-dense n-element point set P in the plane and for any stacking order f, we have*

$$\lim_{\varepsilon \to 0} \text{visible}\bigl(\mathcal{D}(\varepsilon P), f\bigr) \leqslant C' n^{3/4},$$

where C' is a constant depending only on C.