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Packing polyominoes clumsily

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ABSTRACT

For a set D of polyominoes, a packing of the plane with D is a maximal set of copies of polyominoes from D that are not overlapping. A packing with smallest density is called a *clumsy* packing. We give an example of a set D such that any clumsy packing is aperiodic. In addition, we compute the smallest possible density of a clumsy packing when D consists of a single polyomino of a given size and show that one can always construct a periodic packing arbitrarily close in density to the clumsy packing.

1. Introduction

Tiling the plane with polyominoes is a fascinating task occupying the minds of mathematicians, computer scientists, chemists and many others who simply enjoy nice puzzles. For the extensive literature on the topic, see [1,3,6,7,10,11]. Most of the tilings by polyominoes known until recently are periodic. It is believed, see [9] that only in 1994 Penrose gave an example of a small set of polyominoes tiling the plane aperiodically. The mathematical theory of aperiodic tilings lead to research on quasicrystals see [12], that in turn resulted in a Nobel price in chemistry awarded to Shechtman in 2011. Gyárfás, Lehel and Tuza [8] asked about a different type of polyomino arrangements in the plane. They called a set of disjoint polyominoes a *clumsy packing* if no other polyomino can be added without an overlap and the total density is as small as possible. In that paper connected polyominoes of order 2, i.e., dominoes were considered.

Here, we study clumsy packings of general polyominoes. For a more detailed account of the topic, see Walzer [14]. For a few special polyominoes, see Goddard [5]. Next we provide the formal definitions and the statements of our results.

A *cell* is a closed unit square in the plane whose sides are aligned with coordinate axes and whose corners have integer coordinates. A *polyomino* is a finite set of cells. The number of cells in the polyomino is its *area* or *order*. A *connected polyomino* is a polyomino for which the interior of the union of its cells is a connected subset of the plane. Note that a polyomino consisting of two cells that share only a corner is not connected. If one polyomino D' is obtained from another polyomino D by translation, we say that D' is a *copy* of D. Note that we do not consider rotations here. We shall sometimes refer to a polyomino as a union of its cells, and say that a polyomino intersects a subset of the plane if one of its cells intersects this subset. Let $\mathcal{D}$ be a finite set of polyominoes. A *packing of a subset S (of cells) of the plane with $\mathcal{D}$* is a maximal set of pairwise disjoint copies of polyominoes from $\mathcal{D}$ contained in S. Note that here polynominoes are considered as sets of cells, hence two polynominoes in a packing may share an edge or a corner of a cell. Whenever the set $\mathcal{D}$ of polyominoes is clear from the context, we sometimes call a packing with $\mathcal{D}$ simply a packing. Let B_n be the set of cells that form an

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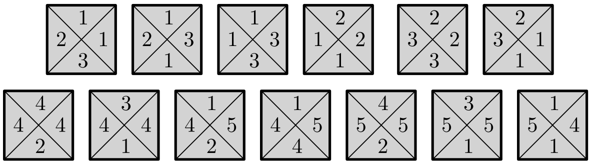

Fig. 1. Set of Wang tiles with edges labeled by integers, [2].

$n \times n$ square either centered at the origin (for even n) or centered at a cell with lower-left corner at the origin. Note that $|B_n| = n^2$. If $\mathcal{P}$ is a packing of the entire plane, its *density* is

$$\mathrm{density}(\mathcal{P}) = \limsup_{n\to\infty} \frac{|(\bigcup_{D\in\mathcal{P}} D) \cap B_n|}{|B_n|}.$$

The density of a packing of a finite area subset S (of cells) of the plane is the fraction of the area occupied by the polyominoes of the packing and the area of S.

For a set of polyominoes $\mathcal{D}$, a packing $\mathcal{P}$ with $\mathcal{D}$ is called *clumsy* if it has the smallest density. We say that a packing $\mathcal{P}$ is *periodic*, if there is a positive integer q such that for any $D \in \mathcal{P}$, $D + (0, q) \in \mathcal{P}$ and $D + (q, 0) \in \mathcal{P}$, where $D + (a, b)$ denotes the translation of D by a vector (a, b). If no such q exists, we say that $\mathcal{P}$ is *aperiodic*. The density of a clumsy packing of the plane with $\mathcal{D}$ is called *clumsiness* of $\mathcal{D}$ and is denoted $\mathrm{clum}(\mathcal{D})$. We prove that the clumsiness is a well-defined notion in a somewhat formal argument, see Appendix A. The main results of this paper are the following four theorems.

Theorem 1. *For any set $\mathcal{D}$ of polyominoes (connected or not) and for any $\epsilon > 0$, there is a periodic packing $\mathcal{P}$ with $\mathcal{D}$ such that* $\mathrm{density}(\mathcal{P}) - \mathrm{clum}(\mathcal{D}) \leqslant \epsilon$.

Theorem 2. *There is a set $\mathcal{D}$ of connected polyominoes such that every clumsy packing with $\mathcal{D}$ is aperiodic.*

The decision problem, see [4], concerning clumsy packings is addressed in the next theorem.

Theorem 3. *The question whether, for a given set $\mathcal{D}$ of connected polyominoes and a given rational number d, $\mathrm{clum}(\mathcal{D}) \leqslant d$, is undecidable.*

Theorem 4. *If $\mathcal{D}$ consists of a single polyomino of order k then $\mathrm{clum}(\mathcal{D}) \geqslant \frac{k}{k^2-k+1} \approx 1/k$. If $\mathcal{D}$ consists of a single connected polyomino of order k then $\mathrm{clum}(\mathcal{D}) \geqslant \frac{k}{k^2-(\lfloor (k-1)/2\rfloor^2+\lceil (k-1)/2\rceil^2)} \approx 2/k$. Both inequalities are tight.*

One of our tools is sets of Wang tiles. A *Wang tile* is a cell with a label assigned to each of its four sides. See Fig. 1 for illustration.

One Wang tile is a *copy* of another Wang tile if one can be obtained from another by translation such that the corresponding labels are preserved. A set $\mathcal{W}$ of Wang tiles *tiles the plane* if the plane is a union of copies of tiles from $\mathcal{W}$ such that any two tiles are either disjoint, share only a corner, or intersect exactly by an edge with the same label. A corresponding set $\mathcal{T}$ of (copies of) Wang tiles is called a *tiling with* $\mathcal{W}$. A tiling $\mathcal{T}$ is *periodic* if there is a positive integer q such that for any tile W in $\mathcal{T}$, $W + (0, q)$ and $W + (q, 0)$ are in $\mathcal{T}$.

Theorem 5. *(See Culik [2].) There is a set of* 13 *Wang tiles that tile the plane and that any such tiling is aperiodic. An example of such a set is shown in Fig. 1.*

2. Wang tiles constructions and properties and difference sets

Given a set $\mathcal{W}$ of Wang tiles, we shall construct a polyomino with parameter x corresponding to each Wang tile $W \in \mathcal{W}$ as shown in Fig. 2. We refer to these polyominoes as $\mathcal{W}$-polyominoes and call the set of all $\mathcal{W}$-polyominoes $\mathcal{D}(\mathcal{W}) = \mathcal{D}(\mathcal{W}, x)$.

Every tiling $\mathcal{T}$ with $\mathcal{W}$ corresponds directly to a packing $\mathcal{P}$ with $\mathcal{W}$-polyominoes, see Figs. 3(a) and 3(b) for an illustration. We have that if tiling $\mathcal{T}$ is aperiodic then $\mathcal{P}$ is aperiodic and

$$\mathrm{density}(\mathcal{P}) = \frac{10x-21}{x^2} = \frac{10}{2l+5} - \frac{21}{(2l+5)^2}. \tag{1}$$

To see that the density of $\mathcal{P}$ is $(10x - 21)/x^2$, observe that the region covered by $\mathcal{P}$ is a periodic set with period x, see Fig. 3(c).

A *difference set of* $\{0, \ldots, q-1\}$ is a subset S of $\{0, \ldots, q-1\}$ such that for every $i \in \{1, \ldots, q-1\}$, there is exactly one ordered pair $(s, s') \in S \times S$ with $i \equiv s - s' \bmod q$. Singer's theorem [13] implies that if n is a prime power then there is a difference set of size $n + 1$ in $\{0, 1, \ldots, n^2 + n\}$. For example, when $n = 2$, $\{1, 2, 4\}$ is a difference set in $\{0, 1, \ldots, 6\}$.