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Triangulations without pointed spanning trees ☆

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Abstract

Problem 50 in the Open Problems Project of the computational geometry community asks whether any triangulation on a point set in the plane contains a pointed spanning tree as a subgraph. We provide a counterexample. As a consequence we show that there exist triangulations which require a linear number of edge flips to become Hamiltonian.

Keywords: Triangulation; Spanning tree; Pointed pseudo-triangulation; Hamiltonian cycle; Edge flip

1. Introduction

Let S be a finite set of points in the plane in general position (no three points are on a common line), and let G be a straight-line graph (drawing in the plane) with vertex set S and edges E. A point $p \in S$ is *pointed* in G if there exists an angle less than π that contains all edges incident to p in G. The graph G is pointed if all its vertices are pointed.

A *triangulation* of S is a maximal planar straight-line graph on the point set S, in the sense that we cannot add an edge without making it non-planar. A planar *spanning tree* of S is a connected, acyclic, plane straight-line graph with vertex set S. Several interesting relations between triangulations and spanning trees exist. For example it is well known that the Delaunay triangulation of S contains a minimum (minimizing the sum of Euclidean edge length) spanning tree of S as a subgraph. Another example is a result of Schnyder [5] who shows that every triangulation of a point set with three extreme vertices allows a partition of its interior edges into three trees.

In this note we disprove the following conjecture which was posed as Problem 10 at the First Gremo Workshop on Open Problems (Stels, Switzerland) in July 2003 (by Bettina Speckmann) and at the CCCG 2003 open-problem

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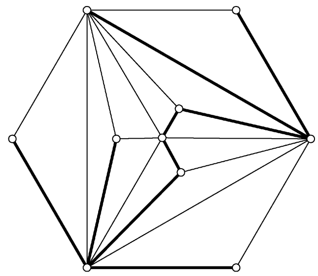

Fig. 1. Triangulation without a spanning path, but containing a pointed spanning tree.

session (Halifax, Canada) in August 2003. It later on became Problem 50 in the Open Problems Project of the computational geometry community [2].

Conjecture 1. *Every triangulation of a set of points in the plane* (*in general position*) *contains a pointed spanning tree as a subgraph.*

This conjecture arose while proving sub-structure properties when investigating flips in pointed and non-pointed pseudo-triangulations [1]. *Pseudo-triangulations* are a generalization of triangulations. A *pseudo-triangle* is a planar polygon with exactly three interior angles less than π. A pseudo-triangulation of S is a partition of the convex hull of S into pseudo-triangles whose vertex set is S. Pseudo-triangulations have become a versatile data structure. Beside several applications in computational geometry, the rich combinatorial properties of pseudo-triangulations have stimulated much research, see, e.g., [1,6] and references therein.

Obviously Conjecture 1 would be true if a triangulation always contained a Hamiltonian path or a pointed pseudo-triangulation as a subgraph. Several triangulations not containing these structures can be found in the literature [4], but for each example it is still easy to find a pointed spanning tree as a subgraph. For an example see Fig. 1, where the triangulation does not contain a spanning path, but a pointed spanning tree (bold edges). This observation supported the general belief that the conjecture should be true. However, in the next section we provide a (non-trivial) counterexample, constructed on a point set S with 124 points. In Section 3 we discuss some implications of this result, like a lower bound for the number of necessary edge flips to transform a given triangulation such that it contains a Hamiltonian cycle.

2. A counterexample

Fig. 2(a) shows the simplest example of a plane connected straight-line graph not containing a pointed spanning tree as a subgraph. We call this graph a 3-*star* and it is a spanning tree which is not pointed at its central point.

The graph defined on the points $a_1, \dots, a_6$ in Fig. 2(b) is called the *bird graph*. It can be seen as two 3-stars plus one edge. In the next lemma we show that the bird graph does not contain a pointed spanning tree either.

Lemma 2. *The bird graph does not contain a pointed spanning tree as a subgraph.*

Proof. Assume that the bird graph contains a pointed spanning tree T as a subgraph. Because of connectivity, the edges a_1a_2 and a_5a_6 are in T. The edge a_2a_5 cannot be in T, as otherwise any edge incident to a_3 would violate pointedness in either a_2 or a_5. Thus, either a_3 or a_4 has to be connected to both, a_2 and a_5. This prevents the other vertex, a_4 respectively a_3, to be connected anyhow. □

In a next step, we extend the bird graph by two additional points b_1, b_2, see Fig. 3. Intuitively speaking b_1 and b_2, respectively, are connected by edges to each visible point of the bird graph. Moreover we add the edge b_1, b_2. We call the resulting full triangulation of the triangle b_1, b_2, a_1 with interior points $a_2, \dots, a_6$ the *cage graph*. All but the three edges forming the outer triangle b_1, b_2, a_1 are called interior edges of the cage graph. The following lemma shows that it will play a crucial role in the construction of a triangulation not containing a pointed spanning tree.